MATHCOUNTS®

2025 State Competition Solutions

Are you wondering how we could have possibly thought that a Mathlete® would be able to answer a particular Sprint Round problem without a calculator?

Are you wondering how we could have possibly thought that a Mathlete would be able to answer a particular Target Round problem in less 3 minutes?

Are you wondering how we could have possibly thought that a particular Team Round problem would be solved by a team of only four Mathletes?

The following pages provide detailed solutions to the Sprint, Target and Team Rounds of the 2025 MATHCOUNTS State Competition. These solutions show creative and concise ways of solving the problems from the competition.

There are certainly numerous other solutions that also lead to the correct answer, some even more creative and more concise!

We encourage you to find a variety of approaches to solving these fun and challenging MATHCOUNTS problems.

Special thanks to solutions author
Howard Ludwig
for graciously and voluntarily sharing his solutions with the MATHCOUNTS community.

Sprint 1

$\frac{\$32}{16\text{ sl}} \times 3\text{ sl} = \frac{\$2}{\cancel{\text{sl}}} \times 3\,\cancel{\text{sl}} = \$\mathbf{6\ or\ \$6.00}$.

Sprint 2

The product of all six faces is $6! = 720$. The *multiside* is 720 divided by the product of the top face times the bottom face—we need to minimize that product to maximize the multiside, which occurs for $1 \times 6 = 6$, not for $2 \times 5 = 10$, nor for $3 \times 4 = 12$. The maximum multiside is thus $720/6 = \mathbf{120}$.

Sprint 3

$\frac{1+5}{3+4+2+1+5+7} = \frac{6}{22} = \frac{\mathbf{3}}{\mathbf{11}}$.

Sprint 4

Only the ones digit of the base 33 plays a role for remainders upon division by 2, 5, and 10. Thus, $33^7 \equiv 3^7 \pmod 5$. The ones digit for powers of 3 go in a cycle of length 4: 3, 9, 7, 1. Thus, 3^7 has the same ones digit as 3^3, namely 7. Any nonnegative integer with a units digit of 2 or 7 upon division by 5 leaves a remainder of **2**.

Sprint 5

$\frac{5 \times 44 \cancel{\times 333} \times 2222 \times 11\,111}{55\,555 \times 4444 \cancel{\times 333} \times 22 \cancel{\times 1}} = \frac{11\,111 \times 2222 \times 44 \times 5}{55\,555 \times 4444 \times 22} = \frac{1 \times 1 \cancel{\times 2 \times 5}}{\cancel{5 \times 2 \times} 1} = \mathbf{1}$.

Sprint 6

$\frac{1}{2} + \frac{1}{3} = \frac{3+2}{6} = \frac{5}{6} = \frac{1}{6/5}$, thus $\mathbf{6/5}$.

Sprint 7

General principle: of all the polygons with a specified number n of sides and specified perimeter p, the one with the greatest area is the regular polygon, each of whose sides has length p/n and each of whose interior angles is $\left(\frac{n-2}{n}\right)180°$. For a triangle of perimeter 24 cm, that means an equilateral triangle with sides $24\text{ cm}/3 = 8\text{ cm}$ and area $\frac{\sqrt{3}}{4}(8\text{ cm})^2 = \mathbf{16\sqrt{3}}\text{ cm}^2$.

Sprint 8

Formulaically, the probability of a roll with sum r is $\Pr(r) = \frac{6-|r-7|}{36}$ for $2 \le r \le 12$, so $\frac{6-0}{36} = \frac{6}{36}$ for $r = 7$ and $\frac{6-3}{36} = \frac{3}{36}$ for $r = 10$ and add. Rationale: There are $6^2 = 36$ arrangements of two regular dice, and the sum of the pips on the two top sides ranges from $1 + 1 = 2$ to $6 + 6 = 12$, with only the one arrangement yielding the given sum for each case. As we successively increment the sum by 1 from 2 up to 7 or decrement the sum by 1 from 12 down to 7, we get one more arrangement yielding the new sum than for the previous sum. Thus, there are 6 arrangement to get a sum of 7 and 3 arrangements to get a sum of 10, for a total of 9 arrangements yielding a sum of 7 or 10 out of the total 36 arrangements, so the desired probability is $9/36 = \mathbf{1/4}$.

Sprint 9

To get to 81 for the average Nana needs on the final test to get the desired 81 plus make up for the loss of 3 points on average for each of the first four tests: $81 + 4 \times (81 - 78) = 81 + 12 = \mathbf{93}$. [NOTE: There is no need to work with total points and the associated bigger numeric values.]

Sprint 10
$d_{\mathrm{MA}} + d_{\mathrm{TH}} = 3\text{ km} + 5\text{ km} = 8\text{ km}$; $d_{\mathrm{MA}} + d_{\mathrm{AT}} + d_{\mathrm{TH}} = 14\text{ km}$. Thus, $d_{\mathrm{AT}} = (d_{\mathrm{MA}} + d_{\mathrm{AT}} + d_{\mathrm{TH}}) - (d_{\mathrm{MA}} + d_{\mathrm{TH}}) = 14\text{ km} - 8\text{ km} = \mathbf{6}\text{ km}$.

Sprint 11
$60 = 2^2 \times 3 \times 5$, so the qualifying factors are: [1] $2 \times 3 = 6$; [2] $2^2 \times 3 = 12$; [3] $2 \times 5 = 10$; [4] $2^2 \times 5 = 20$; [5] $3 \times 5 = 15$. Therefore, there are **5** qualifying divisors.

Sprint 12
$\left(\frac{x^8}{x^3}\right)^3 = (x^{8-3})^3 = (x^5)^3 = x^{(5)(3)} = x^{15}$, so $n = \mathbf{15}$.
[NOTE: Despite the problem wording, this answer applies for any nonzero real value for x.]

Sprint 13
The main diagonals of the cube constitute diameters for the circumscribing circle. The length of such a main diagonal D is $\sqrt{3}s$, where s is the edge length of the cube, which, in turn is the cube root of the volume V_c in the cube. The volume V_s of the inscribed sphere is $\frac{4}{3}\pi r^3$, where r is the radius of the sphere, which is ½ of the diameter D. Putting it all together:

$$V_s = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi\frac{D^3}{2^3} = \frac{\pi}{6}\left(\sqrt{3}s\right)^3 = \frac{\pi}{6} \times 3\sqrt{3}s^3 = \frac{\sqrt{3}\pi}{2}\sqrt[3]{V_c}^3 = \frac{\sqrt{3}\pi}{2}V_c = \frac{\sqrt{3}\pi}{2}(216\text{ cm}^3)$$
$$= 108\pi\sqrt{3}\text{ cm}^3$$, so $a = 108, b = 3$, and $a + b = \mathbf{111}$.

Sprint 14
There are 11 integers from n to $n + 10$ when n is an integer, and their average is $n + 5$, so their sum is $11(n + 5)$, which must be divisible by 10. Because 10 and 11 have no common factors greater than 1, 10 must divide $n + 5$. Because n is a positive integer, $n + 5 \geq 6$. The least integer that is at least 6 and divisible by 10 is 10, so $n + 5 = 10$ and $\frac{11(n+5)}{10} = \frac{11(10)}{10} = \mathbf{11}$.

Sprint 15
Let $a = \mathrm{AE}, b = \mathrm{EB}, h = \mathrm{BC}$. Then $CD = a + b$. The area of the trapezoid is $\big((a + b) + b\big)h/2 = (a + 2b)h/2$. The area of the triangle is $ah/2$. Thus, $\frac{7}{3} = \frac{(a+2b)\cancel{h/2}}{a\cancel{h/2}} = 1 + \frac{2b}{a}$, so $\frac{2b}{a} = \frac{7}{3} - 1 = \frac{4}{3}$, making the desired ratio $\frac{a}{b} = \frac{2}{4/3} = \mathbf{\frac{3}{2}}$.

Sprint 16
$3^9 + 9^3 = 3^9 + 3^6 = 3^6(3^3 + 1) = 3^6(28) = 2^2 \times 3^6 \times 7$, with the greatest prime factor being **7**.

Sprint 17
This is a Pythagorean triples problem. We want the sum of all values h such there is a value l with $h^2 - l^2 = 64\text{ cm}^2$ and each of h/cm and l/cm is a positive integer. If we know the Pythagorean triples well, there is a good chance we would find all values of h, but there is a risk of missing some (more so for a difference of squares being 144 cm^2), so let's use a systematic approach. Our difference of squares is an even number of square centimeters, so h/cm and l/cm must be both even or both odd and their mean m/cm is an integer, so that there is an integer value k/cm with $h = m + k, l = m - k$, and $64\text{ cm}^2 = h^2 - l^2 = (m + k)^2 - (m - k)^2 = 4mk$. Thus, $mk = 16\text{ cm}^2$. Now $l > 0$ cm, so $k < m$. The only choices are $(m, k) = (16, 1)$ cm for which $h = 17$ cm and $(m, k) = (8, 2)$ cm for which $h = 10$ cm, so the desired sum is $17\text{ cm} + 10\text{ cm} = \mathbf{27}$ cm.

Sprint 18
Subtracting the second given equation from the first yields $x - y = y^2 - x^2$, which can be rearranged as $0 = (x^2 - y^2) + (x - y) = (x - y)(x + y + 1)$, implying $x = y$ or $x + y = -1$; however, the given constraints reject $x = y$. Adding the two given equations yields: $x^2 + y^2 = x + y + 8 = -1 + 8 = \mathbf{7}$.

Sprint 19
$b^2 - 5b + 6.21$ is a quadratic equation with two roots. The mean m of the roots is the negative of the linear coefficient divided by twice the leading [quadratic] coefficient, thus $\frac{-(-5)}{2(1)} = \frac{5}{2} = 2.5$. Each root is offset up or down from m by the square root of the difference m^2 minus the quotient of the constant coefficient divided by the leading coefficient, thus, $\sqrt{\left(\frac{5}{2}\right)^2 - \frac{6.21}{1}} = \sqrt{\frac{25}{4} - 6.21} = \sqrt{6.25 - 6.21} = \sqrt{0.04} = 0.2$. For the lesser root, we subtract 2.5 – 0.2 = **2.3**.

Sprint 20
The distance between a point $(X; Y)$ and a line $Ax + By + C = 0$ is given by $\left|\frac{AX+BY+C}{\sqrt{A^2+B^2}}\right|$; the path between the point and the line is perpendicular to the line. The line $\overleftrightarrow{AC}$ has slope $\frac{3-6}{6-2} = -\frac{3}{4}$ and y-intercept $6 - \left(-\frac{3}{4}\right)2 = \frac{15}{2}$; therefore, the slope-intercept equation for $\overleftrightarrow{AC}$ is $y = -\frac{3}{4}x + \frac{15}{2}$, which can be transformed into out desired quasi-standard form as $3x + 4y - 30 = 0$. The distance from B to this line is thus: $\left|\frac{3(2)+4(3)-30}{\sqrt{3^2+4^2}}\right| = \frac{12}{5}$. ABD forms a right triangle with $\overline{AB}$ as the hypotenuse of length $6 - 3 = 3$, $\overline{BD}$ as leg of length $\frac{12}{5}$, and leg $\overline{AD}$ having the length we seek. We have a 3-4-5 right triangle with scale factor $\frac{3}{5}$ and we want the "3" leg, which length is $\frac{3}{5} \times 3 = \mathbf{\frac{9}{5}}$.

Sprint 21
Let E be the point of intersection of the two diagonals, and let $a = \text{AE}$, $b = \text{BE}$, $c = \text{CE}$, $d = \text{DE}$, and $x = \text{BC}$. Then $a^2 + b^2 = (5\text{ in})^2 = 25\text{ in}^2$; $a^2 + d^2 = (11\text{ in})^2 = 121\text{ in}^2$; $c^2 + d^2 = (12\text{ in})^2 = 144\text{ in}^2$; $x^2 = b^2 + c^2$. Adding all three equations together yields $2a^2 + b^2 + c^2 + 2d^2 = 290\text{ in}^2$. Then subtracting 2 times the equation $a^2 + d^2 = (11\text{ in})^2 = 121\text{ in}^2$ yields $x^2 = b^2 + c^2 = 48\text{ in}^2$, so $x = \sqrt{48\text{ in}^2} = \mathbf{4\sqrt{3}}$ in.

Sprint 22
Let the original number be $n = 100h + 10t + u$, where each of h, t, and u is any of the digits 0 through 9. Then the altered number is $n' = 100h + 10u + t = 0.85n = \frac{17}{20}n$. For both n and $\frac{17}{20}n$ to be integers, and because 20 and 17 are relatively prime, n must be divisible by 20, which means that $u = 0$ and t is even, so $t = 2k$ for integer k, $0 \le k \le 4$. Thus, $n = 100h + 20k$ and $n' = 100h + 2k$. Thus, $\frac{3}{20}n = n - n' = 18k$, which means $n = 120k$. Now $k = 0$ must be rejected because n is specified to be positive. Therefore, the sum of all valid values of n is $120(1 + 2 + 3 + 4) = \mathbf{1200}$.

Sprint 23
For any one color, the two marbles can go in AB, AC, AD, BC, BD, or CD—6 options. This is independently true for each of the 3 colors, so there are $6^3 = \mathbf{216}$ total distinct distributions.

Sprint 24

The rightmost digit must be odd and there must be at least two even digits, meaning a total of 3 or 4 digits. Let E represent an even digit (0, 2, 4, 6, 8), N a nonzero even digit (2, 4, 6, 8), O an odd digit (1, 3, 5, 7, 9), and D any nonzero digit (1..9). A 3-digit integer must be of the form NEO for $4 \times 5 \times 5 = 100$ options. A 4-digit integer could be DEEO for $9 \times 5 \times 5 \times 5 = 1125$ options, NEOO for $4 \times 5 \times 5 \times 5 = 500$ options, or NOEO for $4 \times 5 \times 5 \times 5 = 500$ options, yielding a total of $\mathbf{2225}$ options.

Sprint 25

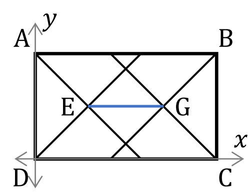
The probability of n heads and $10 - n$ tails is the same as the probability of $10 - n$ heads and n tails due to symmetry associated with fair coins. The desired cases for (H, T) counts are (3, 7) and vice versa (v.v.), (4, 6) and v.v., and (5, 5); undesired cases are (0, 10) and v.v., (1, 9) and v.v., and (2, 8) and v.v. The number of distinct cases is the same either way, but the combinations coefficients are easier to calculate for greater H-T differences, so let's work with complementary probabilities. The probability of n heads and $10 - n$ tails is given by $\frac{10!}{n!\,(10-n)!}\left(\frac{1}{2}\right)^{10}$ and also for n tails. This yields $\frac{1}{1024}$ for each of n = 0 and n = 10; $\frac{10}{1024}$ for each of n = 1 and n = 9; and $\frac{10\times 9}{2\times 1024} = \frac{45}{1024}$ for each of n = 2 and n = 8. Therefore, the complementary scenarios total $2\left(\frac{1}{1024} + \frac{10}{1024} + \frac{45}{1024}\right) = \frac{112}{1024} = \frac{7}{64}$, and the probability of the complementary cases is $1 - \frac{7}{64} = \mathbf{\frac{57}{64}}$.

Sprint 26

$\sqrt{a \pm \sqrt{a^2 - b^2}} = \sqrt{\frac{a+b}{2}} \pm \sqrt{\frac{a-b}{2}}$, for $a \geq |b|$. Here $a = 7$ and $b = 1$, so $\sqrt{7 \pm 4\sqrt{3}} = 2 \pm \sqrt{3}$. Therefore, $\sqrt{7 + 4\sqrt{3}} - \sqrt{7 - 4\sqrt{3}} = \left(2 + \sqrt{3}\right) - \left(2 - \sqrt{3}\right) = 2\sqrt{3}$, so $a = \mathbf{2}$.

Sprint 27

The problem refers to a parallelogram and the length of its sides; however, it neither states nor implies anything about any height or angle measures, so such information must be extraneous. Let's choose the simplest possible parallelogram, namely a 12 cm × 7 cm rectangle to work with. Let's place Vertex D at the origin (0, 0) cm, $\overline{DC}$ along the x-axis, so C is at (12, 0) cm, $\overline{DA}$ along the y-axis with A at (0, 7) cm. Then B is at (12, 7) cm. Each vertex is a right angle, and the angular bisector in each case forms two 45° angles and a line segment with slope 1 or −1. The bisector of ∠A intersects the bisector of ∠D at point E at (3.5, 3.5) cm; the bisector of ∠B intersects the bisector of ∠C at point G at (8.5, 3.5) cm. Thus, EG = 8.5 cm − 3.5 cm = **5** cm.



Sprint 28

"MATHCOUNTS" has 2 Ts, while the other 8 letters are distinct, so the one and only issue is the duplication of T. If at least one T is in slots 1, 2, 9, or 10, we can always choose a substring (the last 8 letters if a T is in slot 1 or 2, and the first 8 letters if a T is in slot 9 or 10) with at most 1 T—it does not matter whether only one T or both Ts are in those four slots. Thus, let's evaluate the complementary case of no Ts in those four slots. The first T can go in any of the 6 slots 3 through 8 out of the total of 10 slots, which has probability $\frac{6}{10} = \frac{3}{5}$; then the second T can go in any of the 5 remaining middle slots out of the total of 9 remaining slots, which has a probability of $\frac{5}{9}$. Therefore, the desired probability is $1 - \frac{3}{\cancel{5}} \times \frac{\cancel{5}}{9} = 1 - \frac{1}{3} = \mathbf{\frac{2}{3}}$.

Sprint 29

$999\,999 = 999 \times 1001 = 3^3 \times 37 \times 7 \times 11 \times 13$, so the sum of all of the factors of 999 999 is $(1+3+9+27)(1+37)(1+7)(1+11)(1+13) = 40 \times 38 \times 8 \times 12 \times 14$. The greatest prime factor of 40 is 5; of 38 is 19; of 8 is 2; of 12 is 3; of 14 is 7. The greatest of these is **19**.

Sprint 30

Expand the two given equations: ① $-xy - 4x + 4y + 16 = 2$; ② $-xy + 4x - 4y + 16 = 3$. Adding ①+② yields $-2xy + 32 = 5$, so $xy = \frac{27}{2}$ and $x^2y^2 = \frac{729}{4}$. Multiplying ①×② yields $6 = [(-xy+16)-(4x-4y)][(-xy+16)+(4x-4y)] = (-xy+16)^2 - (4x-4y)^2 = x^2y^2 \cancel{- 32xy} + 256 - 16x^2 \cancel{+ 32xy} - 16y^2 = x^2y^2 - 16(x^2+y^2) + 256$, so $x^2 + y^2 = \frac{x^2y^2+250}{16} = \frac{729+4\times 250}{64} = \frac{1729}{64}$. Therefore, $(x^2-1)(y^2-1) = x^2y^2 - (x^2+y^2) + 1 = \frac{729}{4} - \frac{1729}{64} + 1 = \frac{11\,664 - 1729 + 64}{64} = \mathbf{\frac{9999}{64}}$.

Target 1

Let g be the greater number and l be the lesser number. Then $g + l = 9876$ and $g - l = 5432$. Though we are seeking g, the two values look easier to subtract than to add while saving time not punching calculator buttons. Subtracting the first equation minus the second yields $2l = 4444$, so $l = 2222$ and $g = 5432 + 2222 = \mathbf{7654}$.

Target 2

The Sundays are 7, 14, 21, and 28, thus 4 of them, leaving 26 days of exercising in April. Thus, the total drunk is $\frac{0.25\text{ L}}{\cancel{10\text{ min}}} \times \frac{\cancel{50\text{ min}}}{\cancel{1\text{ d}}} \times 26\,\cancel{\text{d}} = \frac{1}{4}\text{ L} \times 130 = \mathbf{32.5}$ L.

Target 3

$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32}$ is a geometric series with sum $1 - \frac{1}{32}$. That missing $\frac{1}{32}$ are the Swiss cheeses, of which there are 3, so the total count of cheeses is $32 \times 3 = \mathbf{96}$.

Target 4

Let a and b be the leg lengths of the given right triangle, and c be the hypotenuse. Based on the given area, $ab/2 = 20\text{ cm}^2$, so $b = 40\text{ cm}^2/a$. Thus, by Pythagoras, $c^2 = a^2 + (40\text{ cm}^2/a)^2$ [1]. The perimeter being given as 40 cm means that $c = 40\text{ cm} - a - b = 40\text{ cm} - a - 40\text{ cm}^2/a$, so $c^2 = (-a + 40\text{ cm} - 40\text{ cm}^2/a)^2 = a^2 - 80\text{ cm}\,a + 1680\text{ cm}^2 - 3200\text{ cm}^3/a + (40\text{ cm}^2/a)^2$ [2]. Setting the two expressions [1] and [2] for c^2 equal yields $0 = -80\text{ cm}\,a + 1680\text{ cm}^2 - 3200\text{ cm}^3/a$ after canceling the common terms a^2 and $(40\text{ cm}^2/a)^2$ on both sides of the equals sign. Let's multiply both sides of the equation by $-a/(80\text{ cm})$ to get a quadratic equation in a and remove the common factor of 80 cm in all of the terms: $a^2 - 21\text{ cm}\,a + 40\text{ cm}^2 = 0$. The two roots of this equation are the lengths of the two legs—either solution can be chosen for a and the other solution for b. The important thing is that $a + b$ is given by the negative of the linear coefficient divided by the quadratic coefficient, thus 21 cm, so that $c = 40\text{ cm} - (a+b) = 40\text{ cm} - 21\text{ cm} = \mathbf{19}$ cm.

Target 5

$\left\lfloor 202\,\cancel{\text{str}} \times \frac{1\,\cancel{€}}{6\,\cancel{\text{str}}} \times \frac{1\text{ pie}}{4\,\cancel{€}} \right\rfloor = \lfloor (202/4)/6 \rfloor\text{ pie} = \lfloor 50/6 \rfloor\text{ pie} = \mathbf{8}$ pie.

Target 6

The set has 2025 elements and, thus, 2^{2025} subsets, of which $2^{2025}-1$ are nonempty. Of these, 2^{2024} have greatest element 2025; 2^{2023} have greatest element 2024; 2^{2022} have greatest element 2023; ...; 2^1 have greatest element 2; 2^0 has greatest element 1. Thus, the greatest element has average value $E = (1\times 2^0 + 2\times 2^1 + 3\times 2^2 + \cdots + 2024\times 2^{2023} + 2025\times 2^{2024})/(2^{2025}-1)$. Then $2E = (1\times 2^1 + 2\times 2^2 + \cdots + 2024\times 2^{2024} + 2025\times 2^{2025})/(2^{2025}-1)$. Subtracting $E = 2E - E = \frac{2025\times 2^{2025} - (2^0+2^1+2^2+\cdots+2^{2023}+2^{2024})}{2^{2025}-1} = \frac{2025\times 2^{2025}-(2^{2025}-1)}{2^{2025}-1} = \frac{2024\times 2^{2025}+1}{2^{2025}-1} = 2024 + \frac{2025}{2^{2025}-1}$. Now, $0 < \frac{2025}{2^{2025}-1} < \frac{2048}{2^{2024}} = \frac{2^{11}}{2^{2024}} = \frac{1}{2^{2013}} < \frac{1}{2^1} = \frac{1}{2}$, which rounds to the nearest integer as 0, so E rounded to the nearest integer is **2024**. (The actual value of E is $E = 2024 + 5.256\ldots\times 10^{-607}$, which means the first nonzero fractional digit is 5 in the 607th decimal place. Most calculators used in MATHCOUNTS cannot handle large numbers like 2^{2025} and cannot calculate with enough significant digits to show that the value is not [exactly] 2024. Thus, thinking meaningfully is much more helpful and meaningful than obliviously punching calculator buttons. ☺)

Target 7

The specified conditions indicate $n \equiv 0 \pmod 8$, $n-1 \equiv 0 \pmod 9$, $n-2 \equiv 0 \pmod{10}$, which is equivalent to $n \equiv 0 \equiv -8 \pmod 8$, $n \equiv 1 \equiv -8 \pmod 9$, $n \equiv 2 \equiv -8 \pmod{10}$. Therefore, $n=-8$ is a solution of the system of congruences; however, a constraint requires the least positive integer, since Estrella could not have baked a negative number of cupcakes. The general solution is an offset from −8 by an integer multiple of the least common multiple of the three modulus values 8, 9, and 10: 8 contributes 3 factors of 2; 9 contributes 2 factors of 3; and 10 contributes 1 factor of 5—making the least common multiple $2^3\times 3^2\times 5 = 8\times 9\times 5 = 360$. Therefore, the general solution is $360k-8$ for any integer k. The least possible positive value results from $k=1$, yielding **352**.

Target 8

The probability of getting consecutive coin tosses with the pattern HTT is $\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)\left(\frac{1}{3}\right) = \frac{2}{27}$. The expectation value of the number of tosses needed to first achieve that pattern is the reciprocal of that probability, thus **27/2**.

Team 1

$120\left(1-\frac{1}{3}-\frac{1}{5}-\frac{1}{4}\right) - 16 = 120 - 40 - 24 - 30 - 16 = \mathbf{10}$.

Team 2

The probability that a player achieves the first head on toss k, given probability of head and probability of tail on any one toss is always 1/2 is $\left(\frac{1}{2}\right)^{k-1}\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^k$. For both players to achieve their first head independently on the same toss k is $\left(\frac{1}{2}\right)^k\left(\frac{1}{2}\right)^k = \left(\frac{1}{4}\right)^k$. Thus, the probability that both players achieve their first head on the same toss, regardless of toss number is $\left(\frac{1}{4}\right)^1 + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^3 + \cdots$, which is an infinite geometric series with first term $\frac{1}{4}$ and common ratio $\frac{1}{4}$, which has a sum of $\frac{1}{3}$. Therefore, the probability that the two players achieve their first head on a different toss count is $1-\frac{1}{3} = \frac{2}{3}$. The scenarios that Rowechen will achieve the first head in less tosses versus that Yochen will achieve the first head in less tosses are symmetric and, thus, have equal probability, splitting the $\frac{2}{3}$ into two equal pieces, specifically $\mathbf{\frac{1}{3}}$.

Team 3

Running the same distance takes 1/2 as much time as does walking, but 3 times the distance requires 3 times the time duration: $40 \text{ min} \times \frac{1}{2} \times 3 = \mathbf{60}$ min.

Team 4

The only nonnegative integers that leave a remainder of 0 upon division by each of 6 and 8 are multiples of the least common multiple of 6 and 8, thus $24k$ for any nonnegative integer k. Extending this to common remainders of 0 to 5, inclusive, means that the values of interest are of the form $24k + r$ for integers k and r. We have an additional constraint that these values must be in the range 100 to 200, inclusive, thus restricting $4 \le k \le 8$ (and $4 \le r \le 5$ only for $k = 4$.) Therefore, values satisfying all constraints are 100 and 101 [2 so far], 120 to 125, inclusive [8 so far], 144 to 149, inclusive [14 so far], 168 to 173, inclusive [20 so far], and 192 to 197, inclusive, for a final count of **26**.

Team 5

For a painted cubic stack of $n \times n \times n$ unit cubes, there are 8 unit cubes at the vertices with 3 painted faces, $12(n-2)$ unit cubes along edges except at corners with 2 painted faces, $6(n-2)^2$ unit cubes on the faces except at edges with 1 painted face, and $(n-2)^3$ interior cubes with 0 painted faces. Here $n = 5$, so the expected number of painted unit cube faces is given by $(8 \times 3 + 12 \times 3 \times 2 + 6 \times 3^2 \times 1 + 3^3 \times 0)/5^3 = (24 + 72 + 54 + 0)/125 = 150/125 = \mathbf{6/5}$.

Team 6

There are 8 distinct configurations of box dimensions $(l; w; h)$, each with inner surface area $2[(l+w)h + lw]$: $(36\text{ ft}, 1\text{ ft}, 1\text{ ft}) \to 146\text{ ft}^2$; $(18\text{ ft}, 2\text{ ft}, 1\text{ ft}) \to 112\text{ ft}^2$; $(12\text{ ft}, 3\text{ ft}, 1\text{ ft}) \to 102\text{ ft}^2$; $(9\text{ ft}, 4\text{ ft}, 1\text{ ft}) \to 98\text{ ft}^2$; $(9\text{ ft}, 2\text{ ft}, 2\text{ ft}) \to 80\text{ ft}^2$; $(6\text{ ft}, 6\text{ ft}, 1\text{ ft}) \to 96\text{ ft}^2$; $(6\text{ ft}, 3\text{ ft}, 2\text{ ft}) \to 72\text{ ft}^2$; $(4\text{ ft}, 3\text{ ft}, 3\text{ ft}) \to 66\text{ ft}^2$. The least of these values is $\mathbf{66}\text{ ft}^2$. Which configuration is "best" should not be a big surprise, as it is typical for a closed figure, the ratio of the size of the enclosed region to the size of enclosing boundary (as volume to surface area or area to perimeter) is optimized [maximized] by making the lengths of the relevant dimensions as close to each other as feasible taking into account any given constraints. In this case, the three integers closest to one another yielding a product of 36 are 4, 3, and 3.

Team 7

Two important formulas involving a weighted average a of two values v_1 and v_2 with respective weights w_1 and w_2 are: [1] $a = \frac{w_1v_1 + w_2v_2}{w_1 + w_2} = v_1 + \frac{w_2}{w_1 + w_2}(v_2 - v_1)$; [2] $\frac{w_2}{w_1} = -\frac{v_1 - a}{v_2 - a}$. The v values correspond to the scores by school (Central vs. Western) or by grade level (freshmen vs. sophomores); a corresponds to a "combined" score across schools at one grade level or across grade levels at one school; the w values are proportional to the number of students associated with the corresponding v value. Note that we need not know actual student counts but only values proportional to those counts, and we need not know the constant of proportionality. Let's [totally arbitrarily] assign a w value of 1 to the category of Central HS freshmen—you could pick a different school and/or grade level if you so desire. Let's process Central HS first, with subscripts 1 and 2 being for freshmen and sophomores, respectively, so $v_1 = 71, v_2 = 76, a = 74, w_1 = 1$; then by

[2]: $\frac{w_2}{1} = -\frac{71-74}{76-74} = 1.5$, so for the category of Central HS sophomores, the w value is 1.5. Next, let's process the freshmen, with subscripts 1 and 2 being for Central and Western, respectively, so $v_1 = 71, v_2 = 81, a = 79, w_1 = 1$; then by [2]: $\frac{w_2}{1} = -\frac{71-79}{81-79} = 4$, so for the category of Western HS freshmen, the w value is 4. Next, let's process Western High School, with subscripts 1 and 2 being for the freshmen and sophomores, respectively, so $v_1 = 81, v_2 = 90, a = 84, w_1 = 4$; then by [2]: $\frac{w_2}{4} = -\frac{81-84}{90-84} = \frac{1}{2}$, so for the category of Western HS sophomores, the w value is 2. Finally, we can apply [1] to the sophomores, with subscripts 1 and 2 being for Central and Western, respectively, so $v_1 = 76, v_2 = 90, w_1 = 1.5, w_2 = 2$; then $a = 76 + \frac{2}{1.5+2}(90-76) = 76 + \frac{4}{7}(14) = \mathbf{84}$.

Team 8

For $x = 1$, $1 - 4y + y^2 \geq 0$ and $y \geq 1$ implies $y \geq 2 + \sqrt{3}$, so $4 \leq y \leq 10$ [thus, 7 values].
For $x = 2$, $4 - 8y + y^2 \geq 0$ and $y \geq 1$ implies $y \geq 4 + 2\sqrt{3}$, so $8 \leq y \leq 10$ [thus, 3 values].
For $x = 3$, $9 - 12y + y^2 \geq 0$ and $y \geq 1$ implies $y \geq 6 + 3\sqrt{3}$ exceeds 10, so no additional solutions.
We have 10 values so far. Swapping the role and values of x and y doubles that count to 20 qualifying solutions out of 100 candidate solutions yields a probability of $\frac{20}{100} = \mathbf{\frac{1}{5}}$.

Team 9

Let's set up an analysis involving coordinates. Having five edges all with the same length 3 in and just one edge of length 4 in means that the longest edge is the intersection of two triangular faces with sides 3 in-3 in-4 in, thus isosceles triangles. All other edges have length 3 in, so they form two intersecting equilateral triangles. Place one of the equilateral triangular faces in the xy-plane with its centroid at the origin and one vertex on the positive y-axis. The edge of length 4 in will connect that vertex on the y-axis and the apex of the tetrahedron. Let's label the vertices as A for the apex, B for the base vertex on the positive y-axis, C for the base vertex in Quadrant 4, and D for the base vertex in Quadrant 3. Then $\text{B} = (0; +\sqrt{3}; 0)$ in, $\text{C} = \left(\frac{3}{2}; -\frac{\sqrt{3}}{2}; 0\right)$ in, $\text{D} = \left(-\frac{3}{2}; -\frac{\sqrt{3}}{2}; 0\right)$ in. For vertex A, the only thing we know is that due to the symmetry of the tetrahedron about the plane containing the long edge and the centroid of the base, along with the setup of the coordinate system, is that $\text{A}_x = 0$; for the rest, let's refer to A_y as y and A_z as h (for the height of the tetrahedron). Therefore, until more information is gleaned, $\text{A} = (0; y; h)$. The distance between A and B is 4 in; therefore:

[1] $\left(y - \sqrt{3}\text{ in}\right)^2 + h^2 = (4\text{ in})^2$. The distance between A and C is 3 in; therefore:

[2] $\left(\frac{3}{2}\text{ in}\right)^2 + \left(y + \frac{\sqrt{3}}{2}\text{ in}\right)^2 + h^2 = (3\text{ in})^2$. Expanding and rearranging each of [1] and [2] yields:
[1′] $y^2 - 2\sqrt{3}\text{ in}\,y + h^2 = 13\text{ in}^2$; [2′] $y^2 + \sqrt{3}\text{ in}\,y + h^2 = 6\text{ in}^2$. Subtracting [2] − [1] yields:
$3\sqrt{3}\text{ in}\,y = -7\text{ in}^2$, so $y = -\frac{7}{9}\sqrt{3}$ in. Substituting this value for y in [1] yields

$\left(-\frac{16}{9}\sqrt{3}\text{ in}\right)^2 + h^2 = 16\text{ in}^2$, so $h^2 = \left(16 - \frac{256}{27}\right)\text{ in}^2 = \frac{176}{27}\text{ in}^2$, so $h = \sqrt{\frac{176}{27}}\text{ in} = \frac{4}{3}\sqrt{\frac{11}{3}}$ in. The volume of a tetrahedron is ⅓ of the area of the base times the height. We just found the height h. For the base area, the area of an equilateral triangle of side length s is given by $\frac{\sqrt{3}}{4}s^2$, which for $s = 3$ in yields $\frac{9}{4}\sqrt{3}\text{ in}^2$. Putting all of this together for the volume of the tetrahedron yields

$\frac{1}{3}\left(\frac{9}{4}\sqrt{3}\text{ in}^2\right)\left(\frac{4}{3}\sqrt{\frac{11}{3}}\text{ in}\right) = \sqrt{\mathbf{11}}\text{ in}^3$.

Team 10

The median requires special care, because if the count n of data items is odd, then one of the data values is the median, whereas if n is even, then the median is formed by the average of two data values, which may be the same or different. If n is even and the two middle values are the same (12), then the uniqueness of mode requires there to be at least three 15s. Because the mean is so much greater than the median and mode, the more copies we have of the median and mode, the greater the data values we need to offset the larger count of lesser values; however, the values are constrained to be at most 46, which means the offset needs to involve a mix of greater count of greater values.

Let's try n even, including 11 and 13 for median being 12, and 15 and 15 for mode being 15. We have three values above the median but only one below, so we need to bring in two more below 12, with the greatest available values being 10 and 9. For the mean to be 20, we should have 120 for the sum of six values, but our sum so far is only 73, thus 47 behind. The greatest value we can put in 46 (which reduces the shortfall by 26), but that distorts the median, so we need to put in also the greatest available value less than 12, which is 8 (which enhances our shortfall by 12, so net change of 14 reduction, so only 33 behind). Then put in 45 and 7 to keep the median as is and reduce our shortfall by 12 to 21. Then put in 44 and 6 to keep the median as is and reduce our shortfall by 10 to 11. Then put in 43 and 5 to keep the median as is and reduce our shortfall by 8 to 3. Then put in 39 and 4, thus ending up with 4, 5, 6, 7, 8, 9, 10, 11, 13, 15, 15, 39, 43, 44, 45, 46—16 values with median 12, mode 15, and mean 20.

Now, perhaps we could save by sharing the median and mode values—instead of 11 and 13 for the median and separately 15 and 15 for the mode, but use one of the 15s to participate in both the median and the mode, and we need a 9 to go with the 15 for a median of 12; however, then we have only one value below 12 going against two above 12, so put in an 8 (needs to be below 9 to retain our median of 12). Now we are starting off with a shortfall of 33 for the mean—better than the 47 we started with before, and starting off with two less data values as well. Then put in 46 and 7, which keeps the median at 12 and reduces our shortfall by 13 to 20. Then put in 45 and 6 to reduce the shortfall by 11 to 9. Then put in 44 and 5 to end up with 5, 6, 7, 8, 9, 15, 15, 44, 45, 46—10 values with median 12, mode 15, and mean 20. Thus, we have now gotten down to $n = 10$.

Might we be able to do better with n odd? Then we need to have the desired median value of 12 as an actual data value to be the median and then we need to have the two values of 15 for the mode, but we need two values below 12, thus 11 and 10, to offset the two existing values greater than 12 so that the median can be 12. These five values start us off with a shortfall of 37 for the mean, which is worse than the initial shortfall of 33 in our second trial and we were at only four values—not looking good. Put in 46 and 9, which reduces our shortfall by 15 to 18. Then put in 45 and 8, which reduces our shortfall by 13 to 5. We must put in another pair of values for any reduction in shortfall, which would put us at $n = 11$, thereby losing to our second trial. Therefore, $n = \mathbf{10}$.