2010
MATHCOUNTS STATE COMPETITION

SPRINT ROUND

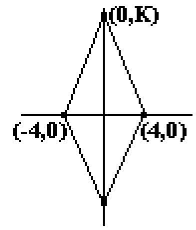
1. We have a mixture of pecans, walnuts and cashews in a ratio by weight of 2:3:1, respectively. 9 pounds of nuts are bought and we are asked to find how many pounds of walnuts are in the mixture.
Let *p* = the number of pounds of pecans.
Let *w* = the number of pounds of walnuts.
Let *c* = the number of pounds of cashews.
Then $w = 3c$ and $p = \frac{2}{3}w = 2c$.
$2c + 3c + c = 9$
$6c = 9$
$c = \frac{9}{6} = \frac{3}{2}$
$w = 3c = \frac{9}{2} = 4.5$ **<u>Ans.</u>**

2. A bookcase has 3 shelves. One shelf contains 10 German books, another 12 math books and a third 8 science books. 6 German books are removed from the bookcase and we are asked to find what fraction of the remaining books are math books.
There are a total of 10 + 12 + 8 = 30 books. 6 German books are moved leaving 4 German books and the 12 math and 8 science books for a total of 4 + 12 + 8 = 24 books.
$\frac{12}{24} = \frac{1}{2}$ **<u>Ans.</u>**

3. What is the sum of the units digits of all multiples of 3 between 0 and 50?
Let's look at the first 10 multiples of 3.
3, **6**, **9**, 1**2,** 1**5**, 1**8**, 2**1**, 2**4**, 2**7**, 3**0.**
The digits are 3, 6, 9, 2, 5, 8, 1, 4, 7, 0. That's all digits from 0 to 9 for a sum of 45.
How many more do we have? (It's certainly not another 10 because that would be 60.) Let's list them.
3**3,** 3**6**, 3**9**, 4**2**, 4**5**, 4**8**
3 + 6 + 9 + 2 + 5 + 8 = 33
45 + 33 = 78 **<u>Ans.</u>**

4. An item was originally priced at $4. Then it's marked up by 100% which means it is now priced at $8. Then it's marked up again by another 200%. That's an additional 8 × 2 = $16.
8 + 16 = 24 **<u>Ans.</u>**

5. The points (4, 0) and (–4, 0) are two non-consecutive vertices of a rhombus with an area of 80. One of the other vertices is (0, K) and we are asked to find K. The two points form one of the diagonals whose length is 8. The rhombus looks like this:



Let's find the length of the other diagonal.
$A = 80 = \frac{1}{2}d_1d_2 = \frac{1}{2}\times 8\times d_2 = 4d_2$
$d_2 = 20$
Half the diagonal is 10 so K = 10 **<u>Ans.</u>**

6. Jan is as old as Gary was 15 years ago. 6 years from now, Gary will be twice as old as Jan so how old is Gary currently?
Let J = Jan's age now.
Let G = Gary's age now.
J = G – 15
2(J + 6) = G + 6
2(G – 15 + 6) = G + 6
2(G – 9) = G + 6
2G – 18 = G + 6
G = 24 **<u>Ans.</u>**

7. 3 flicks equal 8 flecks.
6 flocks equal 4 flecks.
So how many flocks are equivalent to 12 flicks?
Cute.
Let *i* = fl**i**cks.
Let *o* = fl**o**cks.
Let *e*= fl**e**cks.
Then:
$3i = 8e$
$6o = 4e$
$?o = 12i$
$3i = 8e$
$i = \frac{8}{3}e$
$6o = 4e$
$o = \frac{4}{6}e = \frac{2}{3}e$

Since $i = \frac{8}{3}e$ and $o = \frac{2}{3}e$, $i = 4o$.
Therefore, $12i = 48o$ and 48 flocks are equivalent to 12 flicks.
48 **Ans.**

8. The integer m is between 30 and 80 and is a multiple of 6. If m is divided by 5, the remainder is 2. If m is divided by 8, the remainder is 2.
These last two statements say to me that if m is divided by 40, the remainder is 2 because 40 = 8 × 5.
Clearly 42 ÷ 40 = 1 R2.
42 ÷ 8 = 5 R2
42 ÷ 5 = 8 R2
And 42 is a multiple of 6.
42 **Ans.**

9. It takes 4 days for 75 workers to build an embankment. We are asked to find out how many days it would take if only 50 workers were available.
It takes 4 × 75 = 300 worker-days to build the embankment.
$\frac{300}{50} = 6$ **Ans.**

10. Hector spent $200 on shirts and pants. The shirts cost $15 and the pants cost $22 so how many shirts did Hector buy?
Let s = the number of shirts that Hector bought.
Let p= the number of pants.
Then we know that
$15s + 22p = 200$
Uh… we only have 1 equation and but we have 2 unknowns…
Let's try the following. If Hector buys a pair of pants and a shirt, that costs him 15 + 22 = 37
Let's enumerate the multiples of 37 that are less than 200.
37, 74, 111, 148, 185
Subtract each of these from 200.
This amount is either all pants or all shirts.
163, 126, 89, 52, 15
Are any of these multiples of 15 or 22?
15 is a multiple of 15. Nothing is a multiple of 22.
Hector bought 5 pants and 5 shirts for $185. But then he also bought another shirt for $15. The number of shirts he bought was 5 + 1 = 6 **Ans.**

11. How many ounces of pure water must be added to 30 ounces of a 30% solution of acid to yield a solution that is 20% acid?
If a solution is made up of 30% acid it means that for every ounce of the solution, 0.3 ounces are acid. Therefore, there are 0.3 × 30 = 9 ounces of acid and 21 ounces of water in the solution.
Let x = the number of ounces of water we need to add to make a 20% solution of acid (i.e., an 80% solution of water).
$\frac{21+x}{30+x} = 100\% - 20\% = 80\% = \frac{4}{5}$
105 + 5x = 120 + 4x
x = 120 – 105 = 15
15 **Ans.**

12. If only triangles can be used, how many additional triangles must be added to the right side of the third scale so that it is balanced?
Let K be the number of triangles that must be added to the right side of the third scale to balance it. Then 5c = 4t + Kt + 3s, where c = the weight of a circle, t = the weight of a triangle and s = the weight of a square.
From the second scale, we know that 1c = 2t, thus we can say 5c = 10t.
Substituting into 5c = 4t + Kt + 3s, we get 10t = 4t + Kt + 3x, so 6t = Kt + 3s.
Let's now find the relationship between squares and triangles.
Using the first scale: 3s + 2t = 3c. 3s + 2t = 3(2t) = 6t. 3s = 4t.
Substituting the equation 3s = 4t into the equation 6t = Kt + 3s, we have 6t = Kt + 4t and 2t = Kt, so K = 2
Thus we need 2 more triangles on the right hand side to equalize the scale.
2 **Ans.**

13. The sum of the first 20 positive even integers is also the sum of four consecutive even integers. What is the largest of these four integers?
The sum of the first 20 positive even integers is:
2 + 4 + … + 38 + 40 =
42 × 10 = 420
Let x be the odd integer that is one greater than the second consecutive even integer and one less than the third consecutive even integer. Then

$(x - 3) + (x - 1) + (x + 1) + (x + 3) = 420$
$4x = 420$
$x = 105$
The largest of the four consecutive even integers is just 3 more than x.
$105 + 3 = 108$ **Ans.**

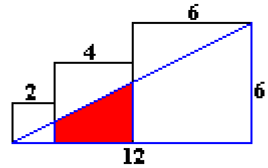
14. We have 6 black chips on the game board as shown.

1	●	3	●
●	6	7	●
9	●	11	12
13	14	●	16

We are asked to find the sum of the numbers in the 2 squares where the seventh and eight chips must be placed so that each row contains exactly 2 chips and each column contains exactly 2 chips.
Clearly the first 2 rows each have 2 chips. The third and fourth do not.
The second and fourth columns have 2 chips and the first and third do not.
So if I place a chip on the 9 I would need to place a chip on the 15 but there's already a chip there. So what if I place a chip on the 11? Then I would have to place a chip on the 13 and all is well. $11 + 13 = 24$ Ans.

15. A set of 7 distinct positive integers has its mean and median both equal to 30. We are asked to find the largest possible integer this set can contain.
If the mean is equal to 30, this means the sum of the 7 numbers is $7 \times 30 = 210$. If the median is also 30, then the fourth number must be 30, since the number of integers in the set is odd.
We are looking for the largest number and we know that all integers are distinct.
Suppose I said the first 3 elements were 1, 2, 3. The fourth is 30. The fifth and sixth should be as close to 30 without being 30 (remember distinct!). So we have:
1, 2, 3, 30, 31, 32 and x.
$x + 1 + 2 + 3 + 30 + 31 + 32 = 210$
$x + 99 = 210$
$x = 210 - 99 = 111$ **Ans.**

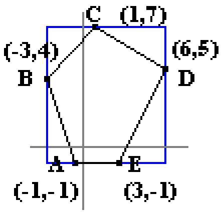
16. We have 3 squares with sides of lengths 2, 4 and 6 arranged side-by side as the figure shows and we are asked to find the area of the shaded quadrilateral.



Note that we have 3 similar triangles (outlined in blue). The largest has 2 sides of 6 and 12. The middle one has one side of $4 + 2 = 6$. Its second side must be proportional to the corresponding side of the larger triangle.
$\frac{6}{12} = \frac{1}{2} = \frac{x}{6}$
So $x = 3$.
The smallest triangle has one side of length 2. Its other side is
$\frac{6}{12} = \frac{1}{2} = \frac{y}{2}$
So $y = 1$.
The area of the middle similar triangle is
$\frac{1}{2} \times 6 \times 3 = \frac{1}{2} \times 18 = 9$
The area of the smallest similar triangle is
$\frac{1}{2} \times 2 \times 1 = 1$
The area of the shaded quadrilateral is just the difference.
$9 - 1 = 8$ **Ans.**

17. The 5^{th} term of a geometric sequence is 11 and the 11^{th} term is 5. So what's the 8^{th} term?
In a geometric sequence, each term is 'x' times the previous term.
11 is the 5^{th} term in the sequence and 5 is the 11^{th} term. Therefore, the 6^{th} term is $11x$, the 7^{th} is $11x^2$, the 8^{th} term is $11x^3$, …, and the 11^{th} is $11x^6$.
$11x^6 = 5$
$x^6 = \frac{5}{11}$
Therefore, $x^3 = \sqrt{\frac{5}{11}}$ and
the 8^{th} term =
$11x^3 = 11\sqrt{\frac{5}{11}} = \sqrt{\frac{5}{11} \times 121} =$
$\sqrt{55}$ **Ans.**

18. The vertices of a convex pentagon are $(-1, -1)$, $(-3, 4)$, $(1, 7)$, $(6, 5)$ and $(3, -1)$. We are asked to find the area of the pentagon.



Normally we would break up the pentagon into smaller triangles but this isn't a really "nice" pentagon. Instead, first enclose the pentagon in a rectangle. Then the four points of the rectangle are (–3, –1) (–3, 7), (6, 7) and (6, –1). This means that the height of the rectangle is 7 – (–1) = 7 + 1 = 8 and the width is 6 – (– 3) =9. Thus, the area is 8 × 7 = 72.

Now we can determine the area of each of the triangles each of which has 2 lines in blue. Name each of the points A, B, C, D and E, respectively. The triangle formed from the points (–3, –1), A and B has its blue lines of size
4 – (–1) = 4 + 1 = 5 and
–1 – (– 3) = –1 + 3 = 2. The area of this triangle is $\frac{1}{2}\times5\times2$ = 5.

The triangle formed by the points (–3, 7), B and C has its blue lines of size
1 – (–3) = 1 + 3 = 4 and
7 – 4 = 3. The area of this triangle is
$\frac{1}{2}\times4\times3$ = 6.

The triangle formed from the points (6, 7), C and D has its blue lines of size
6 – 1 = 5 and
7 – 5 = 2. The area of this triangle is
$\frac{1}{2}\times5\times2$ = 5.

Finally, the triangle formed by the points D, E and (6, –1) has its blue lines of size
6 – 3 = 3 and
5 – (–1) = 5 + 1 = 6. The area of this triangle is
$\frac{1}{2}\times3\times6$ = 9.

Thus, the total area in the rectangle that is **not** in the pentagon is
5 + 6 + 5 + 9 = 25.
72 – 25 = 47 **Ans.**

19. We have two cars, one traveling at 60 km/hr and the other at 75 km/hr. It takes the second car 75 km (from point A) to catch up to the first car (which passed point A prior to the second car). We are asked to find out how many minutes after the first car passed point A, the second car passed point A.
The two cars meet 75 km past point A. At 60 km/hr it took the first car 75 minutes to get there.
But the second car travels at 75 km/hr so it must have been at point A 60 minutes before or 75 – 60 = 15 minutes after the first car had passed Point A.
15 **Ans.**

20. The square of three times a positive integer is decreased by the integer. The result is 2010 and we are asked to find the integer.
Let x = the integer.
$(3x)^2 - x = 2010$
$9x^2 - x - 2010 = 0$
Okay. To factor this equation, let's first get the factors of 2010.
2010 = 3 × 670 = 2 × 3 × 5 × 67
We need a positive value that is one less than the negative value.
67 × 2 = 134
9 × 3 × 5 = 135
$(9x + 134)(x - 15) = 0$
x = 15 **Ans.**

21. Bag A contains 3 white and 2 red balls. Bag B contains 6 white and 3 red balls. If a bag is chosen at random and two balls are drawn from it at random, what is the probability that the two balls are the same color?
The probability of choosing a bag is $\frac{1}{2}$.
There are 5 balls in Bag A and 9 in Bag B.
If Bag A is chosen, then the probability of choosing two white balls is:
$\frac{3\times2}{5\times4} = \frac{3}{10}$
If Bag B is chosen, then the probability of choosing two white balls is:
$\frac{6\times5}{9\times8} = \frac{5}{12}$
Thus, the probability of choosing 2 white balls is:
$\left(\frac{1}{2}\times\frac{3}{10}\right) + \left(\frac{1}{2}\times\frac{5}{12}\right) = \frac{3}{20} + \frac{5}{24} =$
$\frac{18}{120} + \frac{25}{120} = \frac{43}{120}$
If Bag A is chosen, then the probability of choosing two red balls is:

$\frac{2\times1}{5\times4}=\frac{1}{10}$

If Bag B, is chosen then the probability of choosing two red balls is:

$\frac{3\times2}{9\times8}=\frac{1\times1}{3\times4}=\frac{1}{12}$

Thus, the probability of choosing 2 red balls is:

$\left(\frac{1}{2}\times\frac{1}{10}\right)+\left(\frac{1}{2}\times\frac{1}{12}\right)=\frac{1}{20}+\frac{1}{24}=$

$\frac{6}{120}+\frac{5}{120}=\frac{11}{120}$

Therefore, the probability of choosing two balls the same color is:

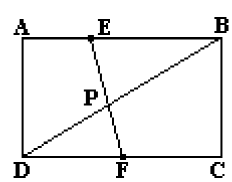
$\frac{43}{120}+\frac{11}{120}=\frac{54}{120}=\frac{27}{60}=\frac{9}{20}$ **Ans.**

22. The base 9 representation of a positive integer is AB and the base 7 representation is BA. We are asked to find the integer expressed in base 10.
AB_9 = (A × 9) + B (in base 10)
BA_7 = (B × 7) + A (in base 10)
9A + B = 7B + A
8A = 6B
$A=\frac{6}{8}B=\frac{3}{4}B$
So what single digit integers for A and B satisfy this equation? If B is 8, then A is 6. If B is 4, then A is 3.
68_9 = (6 × 9) + 8 = 54 + 8 = 62 =
(7 × 8) + 6 = 86_7
Wait a minute! 8 isn't legal in base 7. So we can't use this.
34_9 = (3 × 9) + 4 = 31 = (4 × 7) + 3 = 43_7
31 **Ans.**

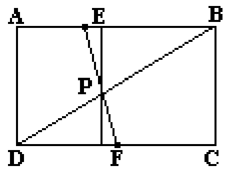
23. For how many positive values of n are both $\frac{n}{3}$ and 3*n*, four-digit integers?
Let's start with 3*n*. 3 × 333 = 999
So we start with 334 as the first multiplier of 3 that gets us 4 digits.
3 × 3333 = 9999 is the last one. So, so far, this means 333 < *n* < 3334
Now consider $\frac{n}{3}$.
The largest value of n has to be $\frac{9999}{3}=3333$ given our range limitations.
The smallest value of n such that $\frac{n}{3}$ is a four-digit integer is 3000 since
$\frac{3000}{3}=1000$
Thus, our final range is 3000 ≤ *n* < 3334 where n is a multiple of 3.
$\frac{3000}{3}=1000$
$\frac{3333}{3}=1111$
That's 1111 – 1000 = 111, but since both 3000 and 3333 are included in the range, we need to add 1. Thus, there are 111 + 1 = 112 values.
112 **Ans.**

24. $3^{x+y}=81$
$81^{x-y}=3$
What is *xy*?
$3^{x+y}=81=3^4$
Therefore, x + y = 4.
$81^{x-y}=\left(3^4\right)^{x-y}=3^{(4x-4y)}=3=3^1$
Therefore, 4*x* – 4*y* = 1
4 × (*x* + *y*) = 4 × 4 = 16
4*x* + 4*y* = 16
4*x* – 4*y* = 1
8*x* = 17
$x=\frac{17}{8}$
$\frac{17}{8}+y=4=\frac{32}{8}$
$y=\frac{32}{8}-\frac{17}{8}=\frac{15}{8}$
$xy=\frac{17}{8}\times\frac{15}{8}=\frac{255}{64}$ **Ans.**

25. In rectangle ABCD, points E and F lie on segments AB and CD, respectively.
AE = $\frac{AB}{3}$ and CF = $\frac{CD}{2}$
Segment BD intersects segment EF at P and we must find the fraction of the area of the rectangle that lies in triangle EBP. The rectangle looks like this:



Let *x* = the length of AB and CD. Then
AE = $\frac{1}{3}x$ and EB = $\frac{2}{3}x$
DF = FC = $\frac{1}{2}x$
Let *y* = the length of AD and BC.
The area of rectangle ABCD is *xy*.
Triangle EPB and triangle DPF are similar to each other since they have the same angles. Draw a line through point P, parallel to AD, and you get the height of both triangles.



The ratio of EB to DF is

$$\frac{\frac{2}{3}x}{\frac{1}{2}x} = \frac{2}{3} \times 2 = \frac{4}{3}$$

Therefore, if h is the height for triangle DPF then $\frac{4}{3}h$ is the height for triangle EPB.

$$h + \frac{4}{3}h = \frac{7}{3}h = y$$

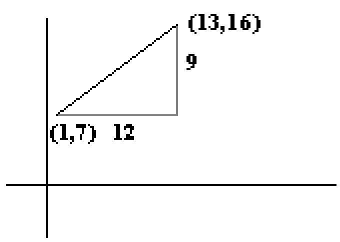
$$h = \frac{3}{7}y \text{ and } \frac{4}{3}h = \frac{4}{3} \times \frac{3}{7}y = \frac{4}{7}y$$

So the area of triangle EPB is

$$\frac{1}{2} \times \frac{2}{3}x \times \frac{4}{7}y = \frac{1}{3}x \times \frac{4}{7}y = \frac{4}{21}x$$

$\frac{4}{21}$ **Ans.**

26. The points (1, 7), (13, 16) and (5, *k*), where *k* is an integer, are vertices of a triangle. We are asked to find the sum of the values of *k* for which the area of the triangle is a minimum.
First, let's plot the 2 points and determine the length of the line we do know about.



If we create a right triangle and make the line connecting (1, 7) and (13, 16) its hypotenuse, then the two sides are 13 – 1 = 12 and 16 – 7 = 9, respectively. This gives us a triangle similar to a 3, 4, 5 right triangle so the length of the hypotenuse is 15.
The next thing is to determine what the *y*-coordinate on the line of length 15 is when the *x*-coordinate is 5. Using (1, 7)
$y = mx + b$
$7 = m + b$
Using (13, 16)
$16 = 13m + b$
$9 = 12m$
$m = \frac{9}{12} = \frac{3}{4}$

$7 = \frac{3}{4} + b$

$b = 7 - \frac{3}{4} = 6\frac{1}{4} = \frac{25}{4}$

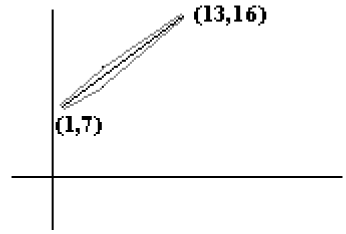
The equation of the line is:

$y = \frac{3}{4}x + \frac{25}{4}$

Now, let's substitute 5 for x.

$y = \frac{3}{4} \times 5 + \frac{25}{4} = \frac{15}{4} + \frac{25}{4} = \frac{40}{4} = 10$

So (5, 10) is on the line. Add or subtract 1 to the *y*-coordinate to get the point that will create the two triangles that have the smallest areas, i.e., (5, 9) and (5, 11). It looks like this:



They're pretty small!
9 + 11 = 20 **Ans.**

27. Solve for x:

$$\sqrt{x + \sqrt{x + \sqrt{x + \ldots}}} = 5$$

This may look hard, but it's not. Just square it.

$$\left(\sqrt{x + \sqrt{x + \sqrt{x + \ldots}}}\right)^2 = 5^2 = 25$$

$$x + \sqrt{x + \sqrt{x + \sqrt{x + \ldots}}} = 25$$

But remember that

$$\sqrt{x + \sqrt{x + \sqrt{x + \ldots}}} = 5$$

$x + 5 = 25$
$x = 20$
Messy, yes, but not hard… 20 **Ans.**

28. What is the smallest possible positive value of $\frac{a+b}{a-b} + \frac{a-b}{a+b}$ if $a > b$?

Note that $\frac{a+b}{a-b}$ and $\frac{a-b}{a+b}$ are reciprocals.

Let $x = \frac{a+b}{a-b}$ in which case $\frac{1}{x} = \frac{a-b}{a+b}$.

Thus, we need to find the smallest positive value of the sum $x + \frac{1}{x}$.

Since we are looking for the smallest positive values of the sum, we only need to consider cases in which x (and $\frac{1}{x}$) are positive. We can also only consider cases in which $x \geq 1$ since for any case in which $0< x <1$, $\frac{1}{x} > 1$. Thus, each such case has an equivalent case in which $x \geq 1$.

Let $x = 1$. Then $x + \frac{1}{x} = 1 + \frac{1}{1} = 2$. If $x \geq 2$, $\frac{1}{x} > 0$ so $x + \frac{1}{x} > 2$.

We are left with the cases $1 < x < 2$. If we graph the equations $y = x$ and $y = \frac{1}{x}$, we see that the equation $y = x$ is increasing faster than the equation of $y = \frac{1}{x}$ is decreasing. Thus, the smallest possible positive values of the sum $\frac{a+b}{a-b}+\frac{a-b}{a+b}$ (and any sum of the form $x + \frac{1}{x}$) is 2.

2 **Ans.**

29. The positive integers A, B and C from an arithmetic sequence. The integers B, C and D form a geometric sequence. If $\frac{C}{B}=\frac{5}{3}$, then what is the smallest possible value of A + B + C + D?

Since $\frac{C}{B}=\frac{5}{3}$, 5B = 3C and C = $\frac{5}{3}$B.

D is a part of the geometric sequence so D = $\frac{5}{3}C=\frac{5}{3}\times\frac{5}{3}C=\frac{25}{9}B$

Since A, B and C are part of an arithmetic sequence and are all integers, there must be an integral difference between them. Let's look at the smallest value of B such that $\frac{5}{3}$B is an integer. B must be, at minimum 3.

Then C = $\frac{5}{3}$B = 5.

That would put A = 1, because 1, 3, 5 is an arithmetic sequence. Will this work? That depends now whether $\frac{25}{9}\times B$ is an integer.

$\frac{25}{9}\times B=\frac{25}{9}\times 3=\frac{25}{3}$

Unfortunately no.

So, now let's try B = 6. Then

C = $\frac{5}{3}$B = 10.

If A = 0, then we would have 0, 5, 10 and this is an arithmetic sequence. Let's compute D again.

$\frac{25}{9}\times B=\frac{25}{9}\times 6=\frac{50}{3}$

No, again! I bet that B = 9 will work!

C = $\frac{5}{3}$B = 15

The difference between C and B is 15 – 9 = 6.

9 = 6 = 3

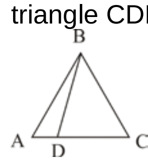
Thus, if A = 3, we have the arithmetic sequence 3, 9, 15.

Compute D again.

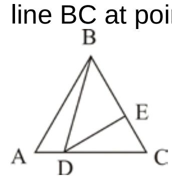
D = $\frac{25}{9}\times B=\frac{25}{9}\times 9=25$

3 + 9 + 15 + 25 = 52 **Ans.**

30. Point D lies on side AC of equilateral triangle ABC so that angle DBC = 45º. We are asked to find the ratio of the area of triangle ADB to the area of triangle CDB.



Drop a perpendicular from angle BDC to line BC at point E.



Then angle DEB is 90º and angle BDE is 45º. So guess what? That gives us an isosceles right triangle. If we call the length of BE x, then DE is also x.

What about triangle DCE? We already know that angle DCE is 60º and since angle DEC is 90º, angle CDE must be 30º.

DE is opposite the 60º angle and we know that the length is x. Let's assume that the length of DC is y. Then

x = $\frac{y\sqrt{3}}{2}$

The length of EC is just $\frac{y}{2}$ because EC is opposite the 30º angle.

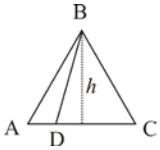
If s = the length of one side of triangle ABC, then

$$s=\frac{\sqrt{3}}{2}y+\frac{1}{2}y=\frac{1}{2}y\left(\sqrt{3}+1\right)$$

$$2s=y\left(\sqrt{3}+1\right)$$

$$y=\frac{2s}{\sqrt{3}+1}$$

Next we need to get the height of triangle ABC. This will let us compute both the area of triangle ABC and the area of triangle BDC since the height is the same value for both.



$h^2+\left(\frac{s}{2}\right)^2=s^2 \rightarrow h^2+\frac{s^2}{4}=s^2$

$h^2=s^2-\frac{s^2}{4}=\frac{3}{4}s^2$

$h=\sqrt{\frac{3}{4}}s=\frac{\sqrt{3}}{2}s$

The area of triangle ABC is

$\frac{1}{2}\times s\times\frac{\sqrt{3}}{2}s=\frac{\sqrt{3}}{4}s^2$

The area of triangle BDC is

$\frac{1}{2}\times y\times\frac{\sqrt{3}}{2}s=$

$\frac{1}{2}\times\frac{2s}{\sqrt{3}+1}\times\frac{\sqrt{3}}{2}s=$

$\frac{1}{2}\times\frac{\sqrt{3}}{\sqrt{3}+1}\times s^2$

The area of triangle ABD is the area of triangle ABC minus the area of triangle BDC.

$\frac{\sqrt{3}}{4}s^2-\frac{1}{2}s^2\left(\frac{\sqrt{3}}{\sqrt{3}+1}\right)$

The ratio of triangle ABD to triangle BDC is:

$$\frac{\frac{\sqrt{3}}{4}s^2-\frac{1}{2}s^2\left(\frac{\sqrt{3}}{\sqrt{3}+1}\right)}{\frac{1}{2}s^2\left(\frac{\sqrt{3}}{\sqrt{3}+1}\right)}=$$

$$\frac{\frac{\sqrt{3}}{4}-\frac{1}{2}\left(\frac{\sqrt{3}}{\sqrt{3}+1}\right)}{\frac{1}{2}\left(\frac{\sqrt{3}}{\sqrt{3}+1}\right)}=$$

$$\frac{\frac{\sqrt{3}}{4}-\frac{2}{4}\left(\frac{\sqrt{3}}{\sqrt{3}+1}\right)}{\frac{2}{4}\left(\frac{\sqrt{3}}{\sqrt{3}+1}\right)}==$$

$$\frac{\sqrt{3}-2\left(\frac{\sqrt{3}}{\sqrt{3}+1}\right)}{2\left(\frac{\sqrt{3}}{\sqrt{3}+1}\right)}=$$

$$\frac{\sqrt{3}}{2\left(\frac{\sqrt{3}}{\sqrt{3}+1}\right)}-1=\frac{\sqrt{3}\left(\frac{\sqrt{3}}{\sqrt{3}+1}\right)}{2}-1=$$

$$\frac{\sqrt{3}+1}{2}-1=\frac{\sqrt{3}+1-2}{2}=$$

$\frac{\sqrt{3}-1}{2}$ **Ans.**

Whew! I'm tired!!!

TARGET ROUND

1. In an archery contest, each person has 2 arrows to shoot into the same circular target. A shot can get anywhere from 1 to 10 points. There are 10 shots that hit the target in an area with different values. Abe got 11 points, Bert got 4, Colin got 7 points, Dave got 16 and Ernie got 17. We are asked to find who hit the part of the target worth 6 points. So we have the values 1, 2, 3, 4, 5, 6, 7, 8, 9, and 10.
   Let's start with Bert. Bert can get 4 points by getting 1 and 3 or 2 and 2. But, wait a minute. Each arrow hits a part of target worth a different value so Bert must have hit 1 and 3.
   That leaves 2, 4, 5, 6, 7, 8, 9, and 10.
   Let's look at Colin. He got 7 points.
   7 = 1 + 6 = 2 + 5 = 3 + 4. Well, 1 and 3 are already gone so Colin must have gotten 2 and 5.
   That leaves 4, 6, 7, 8, 9, and 10.
   Now, let's look at Abe. He got 11 points.
   11 = 10 + 1, 9 + 2, 8 + 3, 7 + 4 and 6 + 5.
   Since 1, 2, 3, and 5 are gone that means Abe must have gotten 4 and 7.
   That leaves 6, 8, 9 and 10
   Dave had 16 points. By inspection that must be 6 and 10.
   That leaves 9 and 8 for Ernie which is fine because 9 + 8 = 17.
   Therefore, Dave hit the area on the target with value 6. Dave **Ans.**

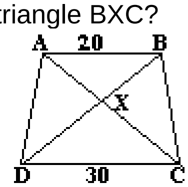
2. How many subsets of the set {M, A, T, H, C, O, U, R, S, E} contain at least one vowel?
   This is equal to the total number of subsets minus the number of subsets that do not contain any vowels. A set with *n* elements has 2^n subsets. Since our set has 10 letters, four of which are vowels and six of which are consonants, there are $2^{10} – 2^6$ = 1024 – 64 = 960 subsets.
   960 **Ans.**

3. We have a 6 oz. pitcher and a 3 oz. pitcher. Each pitcher contains water. The smaller pitcher is $\frac{1}{3}$ full and the

larger one is $\frac{2}{3}$ full. If we fill up the rest of each pitcher with oil and then combine both pitchers, what fraction of the mixture is water?
The smaller pitcher is $\frac{1}{3}$ full with water, i.e., it has 1 ounce of water. We can fill this pitcher with 2 ounces of oil.
The larger pitcher is $\frac{2}{3}$ full with water so, it has 4 ounces of water. We can fill this pitcher with 2 more ounces of oil.
When we combine both pitches we will have 6 + 3 ounces of liquid.
1 + 4 = 5 ounces are water and
2 + 2 = 4 ounces are oil so the mixture is $\frac{5}{9}$ water.
$\frac{5}{9}$ **Ans.**

4. The integer 12 has 5 one-digit factors, 20 has 4 one-digit factors and together 12 and 20 have nine one-digit factors, some of which are repeats. We are asked to find the total number of one-digit factors for all of the integers from 1 to 100, inclusive.
The number 1 is a one-digit factor of all numbers from 1 to 100, inclusive. That's 100 one-digit factors.
The number 2 is a one-digit factor of all even numbers from 1 to 100 or 50 numbers.
100 + 50 = 150
The number 3 is a one-digit factor of all multiples of 3 from 1 to 100. There are 33 of them.
150 + 33 = 183
The number 4 is a one-digit factor of all multiples of 4. There are 25 of them.
183 + 25 = 208
The number 5 is a one-digit factor of all multiples of 5. There are 20 of them.
208 + 20 = 228
The number 6 is a one-digit factor of all multiples of 6. There are 16 of them.
228 + 16 = 244
The number 7 is a one-digit factor of all multiples of 7. There are 14 of them.
244 + 14 = 258
The number 8 is a one-digit factor of all multiples of 8. There are 12 of them.
258 + 12 = 270
The number 9 is a one-digit factor of all multiples of 9. There are 11 of them.
270 + 11 = 281 **Ans.**

5. The mean of the seven numbers x, $2x - 4$, $4x - 3$, -13, 9, $5x + 2$ and $x - 2$ is 4. We are asked to find the integer value of the mode of this set of numbers.
Add up the seven numbers, then divide by 7.
$x + 2x - 4 + 4x - 3 - 13 + 9 +$
$5x + 2 + x - 2 =$
$3x - 4 + 4x - 16 + 5x + 11 + x - 2 =$
$7x - 20 + 6x + 9 = 13x - 11$
Divide by 7 and the value is 4.
$\frac{13x - 11}{7} = 4$
$13x - 11 = 28$
$13x = 28 + 11 = 39$
$x = 3$
Therefore, the values of the seven numbers are:
3, 2, 9, –13, 9, 17 and 1. I see 2 9's. That must be the mode. 9 **Ans.**

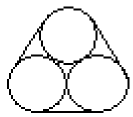
6. Trapezoid ABCD has base AB = 20 and base CD = 30. Diagonals AC and BD intersect at X. The area of trapezoid ABCD is 300 so what is the area of triangle BXC?


The area of the trapezoid is
$\frac{1}{2}(b_1 + b_2)h = 300$
$\frac{1}{2}(20 + 30)h = 25h = 300$
h = 12
The area of BXC can be computed by the formula
$\frac{(b_1 \times b_2)}{2(b_1 + b_2)} \times h = \frac{20 \times 30}{2(20 + 30)} \times 12 =$
$\frac{600 \times 12}{100} = 6 \times 12 = 72$ **Ans.**

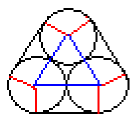
7. How many positive four-digit integers contain the digit grouping "73" at least once?
There are 9 sets of 10 numbers where 7 and 3 are the last 2 digits.
E.g., 1073, 1173, …1993, repeat for the 2000's through 9000's. That's 90.
Then there are 9 sets of 10 numbers where 7 and 3 are the middle 2 digits.

E.g., 1730, 1731, … 1739, repeat for the 2000's through 9000's. That's 90 more. Then there are 100 for 7300 through 7399. That's where the 7 and 3 are the first 2 digits.
That should be it, but first let's consider whether we double counted anything. Yes, we did.
7373 was already counted.
90 + 90 + 100 – 1 = 279 **Ans.**

8. A belt is drawn tightly around three circles of radius 10 cm each. The length of the belt can be written in the form $a + b\pi$ and we are asked to find the value of $a + b$.



The belt that is drawn around the 3 circles is actually composed of 3 straight lines and 3 portions of the circle circumferences.



First connect the centers of the 3 circles (the blue line). This forms an equilateral triangle with sides of size $2r = 20$. Next, from the points of the equilateral triangle (i.e., the centers of the circles), draw lines to the tangents of the circles (in red). These are 90 degree angles and we actually form 3 rectangles of size 20 by 10.
If you look at any circle center, we have the 60º angle formed by the equilateral triangle and 2 90º angles formed by corners of the rectangle. That's a total of 60 + 90 + 90 = 240 degrees. That means that the angle formed by the two red lines meeting at a vertex of the equilateral triangle is 360 – 240 = 120 degrees. And that means that the portion of the circumference delineated by those 2 red lines is just $\frac{1}{3}$ of the circumference.
So, let's take stock. We have 3 straight lines that are just the longer sides of the rectangles or $3 \times 2r = 3 \times 20 = 60$. We have 3 circular parts of the belt each of which is $\frac{1}{3}$ of the circumference of a circle or, aggregating them, one entire circumference of the circle. The circumference of a circle with radius 10 is $2\pi r = 2\pi(10) = 20\pi$
Therefore, the total length of the belt is $60 + 20\pi = a + b\pi$
$a = 60$ and $b = 20$
$a + b = 80$ **Ans.**

Team Round

1. What is the largest five-digit integer whose digits have a product equal to the product (7)(6)(5)(4)(3)(2)(1)?
The largest single digit we can have is 9.
$7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 =$
$7 \times 3 \times 2 \times 5 \times 2 \times 2 \times 3 \times 2 \times 1$
We can remove the both 3's from the expression above. That leaves:
$7 \times 2 \times 5 \times 2 \times 2 \times 2 \times 1$
There are no more digits that we can multiply to get another 9 but there are 3 2s which will give us 8.
So far we have 9 and 8.
We are left with:
$7 \times 5 \times 2 \times 1$
Just primes and 1. So the number must be 98752. We don't need the 1, after all.
The largest five-digit number is 98,752. **Ans.**

2. In triangle ABC, angle A is 86º, angle B is 22º more than three times angle C. We are asked to find the measure of angle C.
Let x = the measure of angle C.
Then the measure of angle B is $3x + 22$.
The sum of all three angles must be 180º.
$86 + 3x + 22 + x = 180$
$108 + 4x = 180$
$4x = 72$
$x = 18$ **Ans.**

3. What is the probability that a particular positive four-digit palindrome is a multiple of 99?
So how many four-digit palindromes are there?
Let's look at 1000-1999. There are: 1001, 1111, 1221, 1331, 1441, 1551, 1661, 1771, 1881 and 1991 for a total of 10. This is true for each range of 1000 numbers so we have $10 \times 9 = 90$ four-digit palindromes.

Now here's what is interesting. All four-digit palindromes are multiples of 11. Let's look at 1001. 1001 = 91 × 11. Each palindrome from 1001 to 1991 is 110 larger than the previous one and 110 is divisible by 11. The first number greater than 2000 that is a palindrome is 2002 and that is 11 larger than 1991, the last number less than 2000 that is a palindrome. So every palindrome is a multiple of 11. All we need to do is to find out which palindromes are multiples of 9. And everyone should know the shortcut here. A number is a multiple of 9 if the sum of all digits in the number is itself a multiple of 9!
So which palindromes qualify?
First of all, no palindrome will have 4 digits that sum to 9. That's because the outer 2 and the inner 2 are the same value so we're really multiplying two numbers by 2. So we have to look to 4 digits that sum to 18, **but** the 4 digits are actually just 2 digits repeated twice. So what do we have?
Just enumerate the two single digits that sum to 9 and you get the answers.
1881, 2772, 3663, 4554, 5445, 6336, 7227, 8118 and 9009.
We must be done, right? No. That's all the palindromes whose digits sum to 18. What about 27? No, that's an odd number. How about 36? There is 1 and that's 9999! Can't forget it!
We have a total of 10 4-digit palindromes that are multiples of 99.
$\frac{10}{90} = \frac{1}{9}$ **<u>Ans.</u>**

4. We want to find the largest integer less than 2010 that has a remainder of 5 when divided by 7, a remainder of 10 when divided by 11 and a remainder of 10 when divided by 13. Note that 10 has a remainder of 10 when divided by both 11 and 13; however, it has a remainder of 3 when divided by 7. Since the least common multiple of 11 and 13 is 143, 10 + 143 = 153 also has a remainder of 10 when divided by both 11 and 13; however, it has a remainder of 6 when divided by 7. We can keep adding 143 until we find a number that has a remainder of 5 when divided by 7. 153 + 143 + 143 = 439 has a remainder of 5 when divided by 7. Since the least common multiple of 7, 11 and 13 is 1001, the next number that satisfies all these criteria is 1440 and the next number after that satisfies the criteria is 2441, which is greater than 2010. Thus, 1440 is the largest integer less than 2010 that has a remainder of 5 when divided by 7, a remainder of 10 when divided by 11 and a remainder of 10 when divided by 13.
1440 **<u>Ans.</u>**

5. The measures of the interior angles of a convex hexagon form an increasing arithmetic sequence. How many such sequences are possible if the hexagon is not equiangular and all of the angle degree measures are positive integers less than 150 degrees?
The number of degrees in a polygon can be computed by the formula
$(n-2)\times 180$ where n is the number of sides of the polygon. Therefore, the number of degrees in a hexagon is
$(6-2)\times 180 = 4\times 180 = 720$
We need arithmetic sequences of 6 elements that sum to 720.
Let's try a difference of 1 in the sequence.
$x + x + 1 + x + 2 + x + 3 + x + 4 + x + 5 = 720$
$6x + 15 = 720$
$6x = 705$
This will not work because x is not an integer. Let's try a difference of 2 between angles.
$x + x + 2 + x + 4 + x + 6 + x + 8 + x + 10 = 720$
$6x + 30 = 720$
$6x = 690$
$x = 115$
This is one possibility. Note that 720 is divisible by 6 (as in $6x$) and 30 is also divisible by 6. What this means is that the additional amount that is added to $6x$ must itself be divisible by 6.
If we use a difference of 3 between the terms in the sequence we get
$6x + 3 + 6 + 9 + 12 + 15 = 6x + 45$.
Okay, there's a pattern here. Using an odd number as the difference in terms will result in a value being added to $6x$ that is not divisible by 6. So we only need consider even values.
Try a difference of 4 between terms.

$x + x + 4 + x + 8 + x + 12 + x + 16 + x + 20 = 720$
$6x + 60 = 720$
$6x = 660$
$x = 110$ (The largest angle is 110 + 20 = 130 so we're still okay.)
Now a difference of 6 between terms.
$x + x + 6 + x + 12 + x + 18 + x + 24 + x + 30 = 720$
$6x + 90 = 720$
$6x = 630$
$x = 105$ (The largest angle is 105 + 30 = 135 so we're still okay.)
Now a difference of 8 between terms.
$x + x + 8 + x + 16 + x + 24 + x + 32 + x + 40 = 720$
$6x + 120 = 720$
$6x = 600$
$x = 100$ (The largest angle is 100 + 40 = 140 so we're still okay.)
Now a difference of 10 between terms.
$x + x + 10 + x + 20 + x + 30 + x + 40 + x + 50 = 720$
$6x + 150 = 720$
$6x = 570$
$x = 95$ (The largest angle is 95 + 50 = 145.)
And now a difference of 12 between terms.
$x + x + 12 + x + 24 + x + 36 + x + 48 + x + 60 = 720$
$6x + 180 = 720$
$6x = 540$
$x = 90$
The largest angle is 90 + 60 = 150 and we violate the requirements.
Therefore, there are 5 sequences that will work.
5 **Ans.**

6. Six-digit integers will be written using each of the digits 1-6 exactly once per six-digit integer. We are asked to find how many different positive integers can be written such that all pairs of consecutive digits of each integer are relatively prime.
Let's list the pairs of integers that are relatively prime. You can use these to create all the six-digit integers.

(1, 2)	(2, 1)	(3, 1)	(4, 1)	(5, 1)	(6, 1)
(1, 3)	(2, 3)	(3, 2)	(4, 3)	(5, 2)	(6, 5)
(1, 4)	(2, 5)	(3, 4)	(4, 5)	(5, 3)	
(1, 5)		(3, 5)		(5, 4)	
(1, 6)				(5, 6)	

The first thing to note is that all pairs must consist of one even and one odd digit. Otherwise, that will leave two even digits together and even numbers are never relatively prime. That will reduce the number of pairs. Also, note that 3 and 6 cannot be used as a pair because 3 divides 6 evenly. This leaves the following pairs.

(1, 2)	(2, 1)	(3, 2)	(4, 1)	(5, 2)	(6, 1)
(1, 4)	(2, 3)	(3, 4)	(4, 3)	(5, 4)	(6, 5)
(1, 6)	(2, 5)		(4, 5)	(5, 6)	

Let's consider the first pair of 2 digits. Then we have 2 more odd integers and 2 more even integers and we will call them o_1, o_2, e_1 and e_2.

Case I: The first digit is odd and the second digit is even.
The only format for the 4 digits that follow is o e o e. (i.e., odd, even, odd, even). Otherwise there will be 2 even digits adjacent to each other. Now, we'll break this case into 3 subcases.

Subcase 1: The first odd digit is not 3 and the first even digit is not 6.
That means that 3 and 6 are in the last four digits and we know that they can not be next to each other.
Let $o_1 = 3$ and $e_2 = 6$. The only format that will satisfy the requirements is
$3\ e_1\ o_2\ 6$
That's 1 combination for this subcase.
The pairs that qualify for this subcase are:
(1, 2), (1, 4), (5, 2) and (5,4).
4 × 1 = **4**

Subcase 2: The first odd digit is not 3 and the first even digit is 6.
This means that 3 is one of the last 4 digits.
Let $o_1 = 3$.
Since the second number is 6, the third, which must be odd, can not be 3; it can only be o_2. That gives us:
$o_2\ e_1\ 3\ e_2$
$o_2\ e_2\ 3\ e_1$
That's 2 combinations for this subcase.
The pairs that qualify for this subcase are:
(1, 6) and (5,6)
2 × 2 = **4**

Subcase 3: The first odd digit is 3 and

the first even digit is not 6.
This means that 6 is one of the last 4 digits.
Let $e_2 = 6$.
Since the 6 is separated from the 3 by at least one digit we can go after all combinations. I.e.,
$o_1\ e_1\ o_2\ 6$
$o_1\ 6\ o_2\ e_2$
$o_2\ e_1\ o_1\ 6$
$o_2\ 6\ o_1\ e_1$
That's 4 combinations for this subcase.
The pairs that qualify for this subcase are (3, 1) and (3, 5).
$2 \times 4 =$ **8**

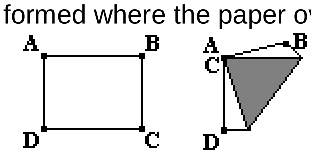
Case II: The first digit is even and the second digit is odd. There are multiple formats for the next 4 digits that can be considered. They are
e o e o
e o o e
o e o e

Subcase 1: The first even digit is not 6 and the first odd digit is not 3.
That means 3 and 6 are in the last four digits and we know that they cannot be next to each other.
Let $o_1 = 3$ and $e_2 = 6$. The following combinations satisfy the requirements:
$6\ o_2\ e_1\ 3$ (for e o e o)
$e_1\ 3\ o_2\ 6$ (for e o o e)
$6\ o_2\ 3\ e_1$ (for e o o e)
$3\ e_1\ o_2\ 6$ (for o e o e)
That's 4 combinations for this subcase.
The pairs that qualify for this subcase are (2, 1), (2, 5), (4, 1) and (4, 5)
$4 \times 4 =$ **16**

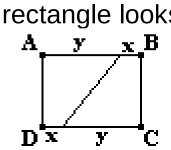
Subcase 2: The first even digit is 6 and the first odd digit is not 3.
This means that 3 is one of the last 4 digits. 6 is, however, separated from the 3 by at least one digit. That means that all possibilities can be considered.
For e o e o there are $2 \times 2 \times 1 \times 1 = 4$ combinations. Similarly, there are 4 combinations for e o o e and o e o e for a total of 12 combinations for this subcase.
The pairs that qualify for this subcase are (6,1) and (6,5).
$2 \times 12 =$ **24**

Subcase 3: The first even digit is not 6 and the first odd digit is 3.
That means 6 is one of the last 4 digits and we know that it can't be next to 3.
Let $e_2 = 6$.
The following combinations satisfy the requirements:
$e_1\ o_1\ 6\ o_2$ (for e o e o)
$e_1\ o_2\ 6\ o_1$ (for e o e o)
$e_1\ o_1\ o_2\ 6$ (for e o o e)
$e_1\ o_2\ o_1\ 6$ (for e o o e)
All 4 combinations of o e o e are also okay because they keep the 6 away from the 3.
That's 8 combinations for this subcase.
The pairs that qualify for this subcase are (2, 3) and (4, 3).
$2 \times 8 =$ **16**
Thus the total number of 6-digit numbers that satisfy the requirements are:
$4 + 4 + 8 + 16 + 24 + 16 =$
$8 + 24 + 40 = 72$ **Ans.**

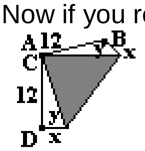
7. A sheet of paper 12-in by 18-in is folded so that two opposite corners touch. What is the area of the shaded triangle formed where the paper overlaps itself?



Here we see the paper unfolded and then folded. After the fold is made, unfold the paper. (Notice the fold runs through the center of the rectangle.) The rectangle looks like this.



The fold has created two segments on AB and CD. Let's call them x and y, respectively. Remember that the sheet of paper is 12×18, so $x + y = 18$.
Now if you refold the paper you see that



the two unshaded triangles that are showing are right triangles with sides of x and 12 and a hypotenuse of y. Remember that $x + y = 18$, thus, $y = 18 - x$
$x^2 + 12^2 = y^2 \rightarrow x^2 + 12^2 = (18 - x)^2$

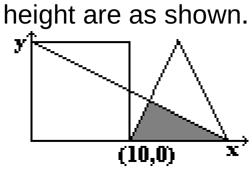
$x^2 + 144 = 324 - 36x + x^2$
$144 - 324 = -36x$
$36x = 324 - 144 = 180$
$x = 5$ and y = 18 – 5 = 13
The area of the two small triangles is, therefore, $2 \times \frac{1}{2} \times 12 \times 5 = 60$.

Note that the shaded triangle covers another portion of the paper which is exactly the same. If we subtract 60 from the area of the paper, we can divide the result by 2 to get the area of the shaded portion. The area of the paper is just
$18 \times 12 = 216$
$216 - 60 = 156$
$\frac{156}{2} = 78$ **Ans.**

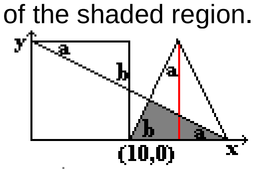
8. A box has a volume of 4320 and surface area of 1704. The sum of the lengths of its 12 edges is 208. We are asked to find the volume of the box if its length, width and height are all increased by one inch.
Let *l, w* and *h* be the length, width and height of the box, respectively.
Then $l \times w \times h = 4320$
And $2lw + 2wh + 2hl = 1704$ or
$lw + wh + hl = 852$
We also know that
$4l + 4w + 4h = 208$ or
$l + w + h = 52$
$(l + 1) \times (w + 1) \times (h + 1) =$
$(lw + l + w + 1) \times (h + 1) =$
$lwh + lh + wh + h + lw + l + w + 1 =$
$lwh + (lh + wh + lw) + (l + w + h) + 1 =$
$4320 + 852 + 52 + 1 = 5225$ **Ans.**

9. Two numbers between 0 and 1 are chosen at random. We are asked to find the probability that the second number chosen will exceed the first number by at least $\frac{1}{4}$.
If we choose the first number greater than $\frac{3}{4}$, the probability is 0 that the second number chosen could be greater than the first number by at least $\frac{1}{4}$. Therefore, the first number must be less than $\frac{3}{4}$. We must choose a number, *x*, between 0 and $\frac{3}{4}$. If we choose 0, then all numbers above $\frac{1}{4}$ will satisfy the requirement. That's $1 - \frac{1}{4} = \frac{3}{4}$ of the available numbers. And if we choose $\frac{3}{4}$, then there is no probability for choosing the second number, i.e., the probability is 0. As the number chosen increases linearly from 0 to $\frac{3}{4}$, the probability decreases linearly from $\frac{3}{4}$ to 0. That makes an average probability of $\frac{1}{2} \times \frac{3}{4} = \frac{3}{8}$. There is a $\frac{3}{4}$ chance of choosing a number in the range of 0 to $\frac{3}{4}$ so the probability is $\frac{3}{4} \times \frac{3}{8} = \frac{9}{32}$. **Ans.**

10. A square and isosceles triangle of equal height are as shown.



The lower right vertex of the square and the lower left vertex of the triangle are at (10, 0). The side of the square and the base of the triangle are of length 10. A segment is drawn from the square's top left vertex to the farthest vertex of the triangle. We are asked to find the area of the shaded region.



In the triangle created within the square by drawing the segment, label the two angles (other than the 90º one as and b). The edge of the square above angle *a* is 10. The line segment at angle *b* is just 5 (because the line is being drawn from (0, 10) to (20, 0) and will cross the vertical line of the square at (10, 5). Let's just stop and figure the hypotenuse of that triangle.
$5^2 + 10^2 = x^2$
$25 + 100 = 125 = x^2$
$x = \sqrt{125} = 5\sqrt{5}$
Okay, next draw the height (in red) of the isosceles triangle. What do you get? Two right triangles with sides 5 and 10 so the hypotenuse must be $5\sqrt{5}$ and not only do we have similar triangles but these 3 triangles are congruent. That means that each of the two triangles

created by drawing the height of the isosceles triangle have angles of measure *a* and *b*, just as shown. Finally, using the corresponding angles theorem, you also know that the angle in the lower right of the shaded triangle has measure *a*.

What does this mean? It means that the shaded triangle is similar (but not congruent) to the triangle within the square and it's a right triangle as well! Now we're getting somewhere. We know the hypotenuse of the shaded triangle is 10. We just need to figure out the sides so we can compute the area. Let x_1 be the side of the shaded triangle that is part of the line segment drawn from the square to the triangle. Then

$$\frac{10}{5\sqrt{5}} = \frac{x_1}{10}$$

$$5\sqrt{5}x_1 = 100$$

$$x_1 = \frac{100}{5\sqrt{5}} = \frac{20}{\sqrt{5}} = \frac{20\sqrt{5}}{5} = 4\sqrt{5}$$

We know that the hypotenuse of this triangle is 10 so let x_2 = the other side of the shaded triangle. Then

$$\left(4\sqrt{5}\right)^2 + x_2^2 = 100$$

$$80 + x_2^2 = 100$$

$$x_2 = \sqrt{20} = 2\sqrt{5}$$

Finally, we can compute the area.

$$\frac{1}{2} \times 2\sqrt{5} \times 4\sqrt{5} = \sqrt{5} \times 4\sqrt{5} =$$

20 **Ans.**