MATHCOUNTS Minis January 2011 Activity Solutions

Warm-Up!
Solve these without writing anything?!?
1. Because 13,225 has a units digit of 5, the number we are squaring must have a units digit of 5. And 13,225 is a bit greater than 10,000, so the number that was squared must be bigger than 100 since $100^2 = 10{,}000$. But is the number 105?115?125? Thinking of 105^2 as $(100 + 5)^2 = 100^2 + (5)(100) + (5)(100) + 5^2$, I can see that it's not quite big enough. With **115**^{2}, I can mentally calculate that it's 10,000 + 1500 + 1500 + 225 = 13,225.

2a. One million is 1,000,000. Granted, we're not supposed to be writing anything, but just looking at this, I can see that $1{,}000{,}000 = 1000^2$, so the square of **999** (or 999^2) would be the largest square less than one million.

2b. This means we need the smallest positive three-digit integer that is 1 more than a multiple of 7. Let's build our smallest positive three-digit multiple of 7... it must be of the form 10_. Dividing 10 by 7 leaves a remainder of 3, and then making the units digit a 5 would make the situation such that 7 now evenly divides into 35. So 105 is the smallest positive three-digit multiple of 7 and **106** is the answer to the original question. (You may have gone a different route... many of us know 77 is a multiple of 7; and if we continue to add 7 or multiples of 7 we can find the number we're looking for. Adding 21 to 77 gives us 98, so 99 would give us a remainder of 1, but isn't a three-digit number; and then adding 7 more we get 106, so this is the smallest positive three-digit integer that is one more than a multiple of 7.)

2c. This is similar to the previous question. I know 999 is a multiple of 9, and that's pretty close to a four-digit number. In fact, if we add 5, we get **1004**, and we know that 1004 will leave a remainder of 5 when it's divided by 9, so this is our answer.

3. Our integer needs to be 3 more than a multiple of 9, and it has to be 3 more than a multiple of 11. This means our integers needs to be 3 more than a multiple of 9 *and* 11 (or 99). Mentally coming up with multiples of 99 could be tough, but 99 × 10 = 990. This is a pretty big three-digit integer, and adding another 99 would make our multiple too big. Adding the remainder of 3 gives us our answer of **993**.

4. We still can't write anything down?!?! I don't know about you, but I think I'm going to have to cheat on this one and hope the video teaches me how to do these mentally!!
Let's start with 993 because we know it's the largest integer that satisfies the second part of the question. However, is it 7 greater than a multiple of 9? 993 – 7 = 986, and this is not a multiple of 9. Let's try 993 – 11 = 982 (which we know still satisfies the second requirement). However, 982 – 7 = 975 is not a multiple of 9.
Let's try 982 – 11 = 971 → 971 – 7 = 964 → not a multiple of 9.
Let's try 971 – 11 = 960 → 960 – 7 = 953 → not a multiple of 9.
Let's try 960 – 11 = 949 → 949 – 7 = 942 → not a multiple of 9.
Let's try 949 – 11 = 938 → 938 – 7 = 931 → not a multiple of 9.
Let's try 938 – 11 = 927 → 927 – 7 = 920 → not a multiple of 9.
Let's try 927 – 11 = **916** → 916 – 7 = 909 → a multiple of 9.
So our answer is **916**.

The Problem is solved in the MATHCOUNTS Mini.

Follow-up Problems

5. Putting words into an equation, as Richard suggested in the video, leads to the equation $7(x^2) + x = 12{,}390$. If we start trying values like 10, 20, 30, ... we see that the results are around 700, 2800 and 6300. We need to keep getting bigger. Let's try 40: 7(1600) + 40 = 11200 + 40 = 11,240. We're getting pretty close. If our integer ends in 1, its square times 7 will end in 7 and then adding a units digit of 1 will get us to a units digit of 8. We need a final units digit of 0. Similar logic tells us that if our integer's units digit is 2, its square times 7 results in a units digit of 8 and then adding a units digit of 2 gets us to a units digit of 0. So let's try **42**. We get $7(42^2) + 42 = 7(1764) + 42 = 12{,}348 + 42 = 12{,}390$.

6. Our equation (or really inequality) can be written as $x + x^2 < 8000$. Sticking with x being multiples of 10, we can see that x^2 is 100, 400, 900, 1600, 2500, 3600, 4900, 6400 and 8100. Our answer must be a bit less than 90. Trying 89, we get $89 + 89^2$. Keep in mind that $89^2 = (90 - 1)^2$, which is $90^2 - 90 - 90 + 1 = 8100 - 180 + 1 = 7921$. This means $89 + 89^2 = 89 + 7921 = 8010$. Seeing how changing our number from 90 to 89 resulted in a difference of 90, it's pretty safe to say that going down to x = **88** will result in a number less than 8000. (Testing it, we see $88 + 88^2 = 7832$.)

Another solution with a similar beginning is to see that once we've determined x must be a bit smaller than 90, we can factor the left side of our inequality to get $x(x + 1) < 8000$. This means we're looking for the lesser of two consecutive integers that have a product just under 8000. Trying (90)(89), we see that's one less 90 than (90)(90), so the product is 8100 – 90 = 8010... not small enough. Let's try (89)(88), which is two fewer 89s than (90)(89), so we have 8010 – 89 – 89 = 7832, and the lesser of the two consecutive integers is **88**.

7. The information we need to work with is shown in the table. Similar to the solution in the video, we see that dividing by either 9 or 13 will result in a remainder of 7. Based on this information, our number could be 7, 7 + (9)(13), 7 + (9)(13) + (9)(13), etc. The value of (9)(13) is 117. When 117 is divided by 5, the remainder is 2.

Divide by	Remainder
5	3
9	7
13	7

	7 +	(9)(13) +	(9)(13) +	(9)(13) +	(9)(13) +	...
Remainder when divided by 5		2	2	2	2	2 ...

From this we see that 7 + (9)(13) has a remainder of 4 when divided by 5. Then 7 + (9)(13) + (9)(13) has a remainder of 6 when divided by 5, which means a remainder of 1. Then 7 + (9)(13) + (9)(13) + (9)(13) has a remainder of 8 when divided by 5, which means a remainder of 3, and this is our desired value. 7 + (9)(13) + (9)(13) + (9)(13) = 7 + 117 + 117 + 117 = **358**.

8. Whoa! This seems way more complicated. Let's look at our scenario in a table. Though it seems to have many more requirements, notice that some of them are repetitive. For instance, if a number leaves a remainder of 3 when divided by 4, it will also leave a remainder of 1 when divided by 2, so we really don't need the first row of our table. Similarly, a number that leaves a remainder of 5 when divided by 6 will also leave a remainder of 2 when divided by 3, so the second row of our table is not necessary

Divide by	Remainder
2	1
3	2
4	3
5	4
6	5
7	6

either. Take a look at the scenarios of dividing by 4 and dividing by 6. These divisors have some common factors. The numbers 3, 7, 11, 15, ... satisfy the conditions for dividing by 4. The numbers 5, 11, 17, 23, ... satisfy the conditions for dividing by 6. The number 11 is in both lists, so satisfies both conditions. So will 11 plus any multiple of 12 (which is the least common multiple of 4 and 6. The requirements for dividing by 4 and 6 can be rephrased as finding a number that has a remainder of 11 when divided by 12. So now we have the scenario shown to the right. This seems a bit more manageable. The first row tells us our number will have a units digit of 4 or 9. Using the second row, we see 6, 13, 20, 27, 34, ... work. Using the third row, we see 11, 23, 35, 47, work. There isn't any overlap yet, so let's keep going:

Divide by	Remainder
5	4
7	6
12	11

6, 13, 20, 27, 34, 41, 48, 55, 62, 69, 76, 83, 90, ...
11, 23, 35, 47, 59, 71, 83, 95, 107, 119, 131, 143, ...
So now we know 83 works for the last two rows, and so will 83 + (7)(12), 83 + (7)(12) + (7)(12), 83 + (7)(12) + (7)(12) + (7)(12), etc. We need to find which of these have a units digit of 4 or 9. Since (7)(12) = 84, we can simplify this list of possibilities to 83, 167, 251, 335, **419**, ...

Want to see an easier solution?? In the original table we can see that if our desired number was just 1 more than it is, it would be *exactly* divisible by 2, 3, 4, 5, 6 and 7. This means our number is 1 less than the least common multiple of 2, 3, 4, 5, 6 and 7. Notice that 12 is the least common multiple of 2, 3, 4 and 6. So our number is (5)(7)(12) – 1 = 420 – 1 = **419**.

Further Exploration

9. **No**. One counterexample is if Johnny's remainder is 0 and Jane's remainder is 1. The mystery number must be a multiple of 3 to satisfy Johnny's remainder, but then it couldn't have a remainder of 1 when divided by 9.

10. **Yes**.

11. If the numbers by which Johnny and Jane divide are relatively prime, then they will be able to find a number.