MATHCOUNTS Minis November 2010 Activity Solutions

Warm-Up!

1. Because these are each arithmetic sequences, we know the difference between consecutive terms for each sequence remains constant. (In other words, the same amount is added to each term to get the next term.)

(a) 11 – 5 = 6, so the common difference is 6. The terms are 5, 11, *11 + 6 or* **17**, *17 + 6 or* **23**, *23 + 6 or* **29**.

(b) Again, the common difference is 6, but we must work backwards. __, __, *5 – 6 or* **–1**, 5, 11. Then __, *–1 – 6 or* **–7**, **–1**, 5, 11. And finally *–7 – 6 or* **–13**, **–7**, **–1**, 5, 11.

(c) Going from 5 to 11, we must add the common difference to 5 a total of four times. The total difference is 6. Dividing this into four equal parts, we see the common difference of the arithmetic sequence is 6/4 = 3/2 = 1.5. Therefore, the sequence is 5, **6.5**, **8**, **9.5**, 11.

2. Because these are each geometric sequences, we know the ratio of consecutive terms for each sequence remains constant. (In other words, the same amount is multiplied by each term to get the next term.)

(a) 18 ÷ 2 = 9, so the common ratio is 9. The terms are 2, 18, *18 × 9 or* **162**, *162 × 9 or* **1458**, *1458 × 9 or* **13,122**.

(b) Again the common ratio is 9, but we must work backwards. __, __, *2 ÷ 9 or* **2/9**, 2, 18. Then __, *2/9 ÷ 9 or* **2/81**, **2/9**, 2, 18. And finally, *2/81 ÷ 9 or* **2/729**, **2/81**, **2/9**, 2, 18.

3. Knowing the first two terms of the arithmetic sequence are 5, 2 tells us the common difference is 2 – 5 = –3 (or we subtract 3 to get from one term to the next). Therefore, the third and fourth terms of the sequence are 2 – 3 = –1 and –1 – 3 = –4, respectively. Now we know –1, –4 are the first two terms of a geometric sequence. The common ratio is –4 ÷ –1 = 4 (or we multiply by 4 to get from one term to the next). Therefore the third term of the geometric sequence is –4 × 4 = –16 and the fourth term is –16 × 4 = **–64**.

4. Knowing the first two terms of the arithmetic sequence are 4, 10 tells us the common difference is 10 – 4 = 6. To get from 4 to 1000, we will need to add 1000 – 4 = 996. This means we will have to start with 4 and add 6 a total of 996 ÷ 6 = 166 times. This will get us to exactly 1000. We will first exceed 1000 by starting with 4 and adding 6 a total of 167 times. The result is 4 + 6(167) = **1006**.

The Problem is solved during the MATHCOUNTS Mini.

Follow-up Problems

5. Notice here that we know the 5th and 21st terms of a geometric sequence of positive numbers, and the term we are looking for is the 13th term, which is the one half-way between the two known terms. We could use a solving method similar to the one shown for the first problem in the Mini, or we can see that it's again true that our missing value is the square root of the product of the two known values: $\sqrt{(7 \times 28)} = \sqrt{(7 \times 7 \times 2 \times 2)} =$ **14**.

6. Consider that to get from the third term to the sixth term, you would need to multiply by the common ratio 3 times. In other words, the third term is –2, and we are looking for the sixth term, which is $(-2)(r^3)$. We also know $(-2)(r^{12}) = -162$. Since the exponents of r are both multiples of 3, and we'd rather work with smaller numbers, what if we let $y = r^3$? Then $(-2)(r^{12}) = -162$ becomes $(-2)(y^4) = -162$. Dividing both sides of the equation by –2 gets us

$y^4 = 81$. And then taking the fourth root, which some people may recognize and is easier than taking a 12th root, we see $y = \pm 3$. Remember we are looking for $(-2)(r^3)$ and $y = r^3$, so we want $(-2)(\pm 3)$. Since all of our terms are negative, we'll use $r^3 = 3$, and the sixth term is **-6**.

7. For Joan and John, their geometric sequences can be written 6, __, 2. Using r as the common ratio, we see the sequence(s) can be rewritten as 6, $6r$, $6r^2$. This means $6r^2 = 2$ or $r^2 = 1/3$. When taking the square root of each side, we see $r = \pm\sqrt{(1/3)} = \pm\sqrt{3}/3$. Since we weren't specifically told this was a geometric sequence of positive numbers, Joan and John could be using different ratios, and they are ***not*** necessarily thinking of the same sequence. One sequence could be 6, $2\sqrt{3}$, 2 and the other could be 6, $-2\sqrt{3}$, 2.

8. This situation represents a geometric sequence with a common ratio of 2. (The number of amoebas for the first few days is 1, 2, 4, 8, 16, ...) Using the formula from the Mini (nth term = ar^{n-1}), on the 23rd day, there would be $(1)(2^{22}) = 2^{22}$ amoebas. If 2^{22} amoebas are required to fill the puddle, then $2^{22} \div 2 = 2^{21}$ amoebas are required for the puddle to be half-full. Notice $n - 1 = 21$, so $n = 22$, and we see it would happen on the **22nd day**. This makes sense since the number of amoebas doubled every day... the puddle would be half-full on the day before it was full!

Further Exploration

9. If 1, a, b is an arithmetic sequence, the common difference is $a - 1$, and the sequence could be rewritten as 1, $1 + (a - 1)$, $(1 + (a - 1)) + (a - 1)$. This is equivalent to 1, a, $2a - 1$. If 1, a, b is a geometric sequence, the common ratio is $a \div 1$, and the sequence could be rewritten as 1, $1 \times a$, $(1 \times a) \times a$. This is equivalent to 1, a, a^2. From the two ways the equation can be rewritten, we now have $2a - 1 = a^2$. Setting the equation equal to 0, we get $0 = a^2 - (2a - 1)$ or $0 = a^2 - 2a + 1$. This can be factored to show $0 = (a - 1)^2$. Thus the only solution for a is 1. The answer is **yes**.

10. Let's break this down into steps.
Step I. If x, y, z is a geometric sequence with common ratio r, then x, y, z can be rewritten as x, xr, xr^2. This means $y = xr$, $z = yr$ and $z = xr^2$.

Step II. If x, $2y$, $3z$ is an arithmetic sequence, the common difference is $2y - x$, so $3z = 2y + (2y - x) = 4y - x$.

Step III. Taking this last equation $3z = 4y - x$, we can use the values from Step I and substitute in for z and y putting all terms in terms of x and/or r. This means $3z = 4y - x$ becomes $3xr^2 = 4xr - x$.

Step IV. Let's solve the quadratic equation setting it equal to 0. We get $0 = 3xr^2 - 4xr + x$. Factoring out the x, we have $0 = x(3r^2 - 4r + 1)$. Factoring the trinomial gets us to $0 = x(3r - 1)(r - 1)$. Now we see either $x = 0$, $3r - 1 = 0$ or $r - 1 = 0$. Therefore, $x = 0$, $r = 1/3$ or $r = 1$. Notice that if $x = 0$, then every term in the geometric sequence would be 0 and $x = y$. We were told $x \neq y$. Similarly, if $r = 1$, then all of the terms in the geometric sequence would be the same and $x = y$. The only value for r is then **1/3**. Notice this makes the common difference for the arithmetic sequence $(-1/3)x$.