May 2010 Activity Solutions

Warm-Up!

1. Each minute, Tina goes up 3 stairs and down 2 stairs, which means she goes up a total of 1 stair per minute. So, in 40 minutes, Tina goes up 1 × 40 = **40** stairs.

2. The 4 from the first term and the −4 from the second term sum to 0. The 5 from the second term and the −5 from the third term sum to 0. The 6 from the third term and the −6 from the fourth term sum to 0. This pattern of canceling out will continue through to the last term, until we are left with only −3 from the first term and 2010 from the last term. Therefore, the sum is −3 + 2010 = **2007**.

3. The 5 in the numerator of the first term and the 5 in the denominator of the third term cancel each other out (because 5/5 = 1, and multiplying by 1 does not change the product). The 6 in the numerator of the second term and the 6 in the denominator of the fourth term cancel each other out. This pattern of canceling out will continue through to the last term, until we are left with only the first two denominators (3 and 4) and the last two numerators (2009 and 2010). Thus, the product is (2009 × 2010) ÷ (3 × 4) = 4,038,090 ÷ 12 = **336,507.5**.

4. (a) 1/(1 × 2) + 1/(2 × 3) = 1/2 + 1/6 = 3/6 + 1/6 = 4/6 = **2/3**
(b) 1/(1 × 2) + 1/(2 × 3) + 1/(3 × 4) = 2/3 + 1/12 = 8/12 + 1/12 = 9/12 = **3/4**
(c) 1/(1 × 2) + 1/(2 × 3) + 1/(3 × 4) + 1/(4 × 5) = 3/4 + 1/20 = 15/20 + 1/20 = 16/20 = **4/5**
Following this pattern, we can see that the final sum will be **2009/2010**.

The Problems are solved in the MATHCOUNTS® Mini video.

Follow-up Problems

5. Using the method in the Mini, we can show that $\frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)}$, so the sum given in Problem 4 is also a *telescoping series*. Then,

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \ldots + \frac{1}{2009\cdot 2010} =$$

$$(1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{4}) + \ldots + (\frac{1}{2009} - \frac{1}{2010})$$

The −1/2 from the first term and the 1/2 from the second term sum to 0. The −1/3 from the second term and the 1/3 from the third term sum to 0. This pattern of canceling out will continue through to the last term, until we are left with only 1 from the first term and −1/2010 from the last term. Therefore, the sum is 1 + (−1/2010) = **2009/2010**. Thus, our guess to Problem 4 is correct.

6. As the denominators increase, the numbers added to the sum approach 0, so the infinite sum is **1.5**.

7. The value of n cannot be 25 or less. If this were the case, the negative values of -1 through $-n$ would sum to 0 with the positive values of 1 through n and would thus cancel out. If n were less than 25, the total sum would remain negative. If n were 25, the total sum would be 0. Therefore, the smallest integer n for which the sum is at least 26 is 26. The values of -25 through -1 sum to 0 with the values 1 through 25, so when $n = 26$, the sum is 0 + 26 = **26**.

8. Let's start by looking at the given terms individually. The first term is $3/(1^2 \times 2^2) = 3/4$. The second term is $5/(2^2 \times 3^2) = 5/(4 \times 9) = 5/36$. The third term is $7/(3^2 \times 4^2) = 7/(9 \times 16) = 7/144$. Now, we can start adding these terms together. The first term on its own is <u>3/4</u>. The sum of the first and second terms is 3/4 + 5/36 = 27/36 + 5/36 = 32/36 = <u>8/9</u>. The sum of the first, second and third terms is 8/9 + 7/144 = 128/144 + 7/144 = 135/144 = <u>15/16</u>. From this pattern, we can find that the sum including the fourth term is <u>24/25</u>; the sum including the fifth term is <u>35/36</u>; etc. Thus, the sum of this infinite series is **1**.