MATHCOUNTS® Minis October 2009 Activity Solutions

Warm-Up!

1. In a race with three people where no ties are allowed, there are $3! = 3 \times 2 \times 1 =$ **6** ways they can finish the race.

2. In a race with four people where no ties are allowed, there are $4! = 4 \times 3 \times 2 \times 1 =$ **24** ways they can finish the race.

3. When there are no ties allowed, a race with n people can finish the race in $n!$ different ways.

4. If ties are allowed in a two-person race, Person A could win, Person B could win, or Person A and Person B could tie. Thus, there are **3** possible orders in which they could finish the race.

The Problems are solved in the MATHCOUNTS® Mini video.

Follow-up Problems

5. We must consider two scenarios: (1) there are no ties, and (2) two participants finish in a tie. First, if there are no ties between any of the three participants, there are $3! = 3 \times 2 \times 1 = \underline{6}$ ways in which they could finish the race. Next, it would be possible for there to be an Alfred/Brandon tie, a Brandon/Charles tie and an Alfred/Charles tie. Each of these three ties could be a tie for first place or a tie for last place, so there are $3 \times 2 = \underline{6}$ ways in which two of the participants could tie. Thus, there are $6 + 6 =$ **12** ways this race could finish.

6. With four participants, there are several cases to consider.

<u>Case 1</u>: There are no ties. With no ties, there are $4! = 4 \times 3 \times 2 \times 1 = \underline{24}$ ways they can finish the race.
<u>Case 2</u>: There is one two-way tie. In this case, there are 6 possible tie pairs: Alfred/Brandon, Alfred/Charles, Alfred/David, Brandon/Charles, Brandon/David, Charles/David. Let's treat each tie pair as a single unit, called T. We'll refer to the other two participants, who are not in a tie, as X and Y. The possible finishing orders are TXY, TYX, XTY, YTX, XYT, YXT. So, with 6 tie pairs and 6 finishing orders with these ties, there are $6 \times 6 = \underline{36}$ ways they could finish the race with one two-way tie.
<u>Case 3</u>: There are two two-way ties. There are three sets of two pairs: (Alfred/Brandon, Charles/David), (Alfred/Charles, Brandon/David), (Alfred/David, Brandon/Charles). Either pair could finish first or last, so there are $3 \times 2 = \underline{6}$ ways for the race to end with two two-way ties.
<u>Case 4</u>: There is a three-way tie. There are four possible triples: (Alfred, Brandon, Charles), (Alfred, Charles, David), (Alfred, Brandon, David), (Brandon, Charles, David). The fourth participant, who is not in the tie, could finish either before or after the other three participants, so there are $4 \times 2 = \underline{8}$ ways for the race to finish with a three-way tie.

Case 5: There is a four-way tie. There is only 1 way for this to happen.

Thus, there are 24 + 36 + 6 + 8 + 1 = **75** possible ways they could finish the race.

7. The first 15 Fibonacci numbers are 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610.

8. Lou could climb only one stair with each step. There is only 1 way to do this. Alternatively, Lou could take one stair for two steps and two stairs for one step. There are 3 ways to do this: (2, 1, 1), (1, 2, 1), (1, 1, 2). Finally, Lou could take two stairs for two steps, and there is only 1 way to do this. So, there are 1 + 3 + 1 = **5** ways to climb the flight of stairs.

9. When there are 5 stairs, there is 1 way to climb one stair with each step. There are 4 ways to climb two stairs with one step and one stair for the remaining steps: (1, 1, 1, 2), (1, 1, 2, 1), (1, 2, 1, 1), (2, 1, 1, 1). There are 3 ways to climb two stairs for two steps and one stair with the remaining step: (1, 2, 2), (2, 1, 2), (2, 2, 1). Thus, there are 1 + 4 + 3 = **8** ways for Lou to climb 5 stairs. Following this pattern, there are **13** ways to climb 6 stairs, **21** ways to climb 7 stairs and **34** ways to climb 8 stairs. These are the Fibonacci numbers.

10. The next-to-last step on a staircase of $n + 1$ stairs can finish at 1 or 2 stairs below the top. One stair below the top is n stairs, so there are $f(n)$ ways to climb to this point. Two stairs below the top is $n - 1$ stairs, so there are $f(n - 1)$ ways to climb to this point. So, the number of ways to climb $n + 1$ stairs is the number of ways to get to 1 stair below the top plus the number of ways to get to 2 stairs below the top, or $f(n + 1) = f(n) + f(n - 1)$.

11. $r(2) = \binom{2}{1}r(1) + 1 = 2(1) + 1 = \mathbf{3}.$ This answer is the same as in problem 4.

12. $r(3) = \binom{3}{1}r(2) + \binom{3}{2}r(1) + 1 = 3(3) + 3(1) + 1 = \mathbf{13}.$ This answer is the same as in the problem in the video.

13. $r(4) = \binom{4}{1}r(3) + \binom{4}{2}r(2) + \binom{4}{3}r(1) + 1 = 4(13) + 6(3) + 4(1) + 1$

$= 52 + 18 + 4 + 1 = \mathbf{75}.$

This answer is the same as in problem 6.

14. In this recursion, all possible scenarios of how a race can finish with any number of participants, *n*, are accounted for:

$$\binom{n}{1}r(n-1) + \binom{n}{2}r(n-2) + \ldots + \binom{n}{n-1}r(1) + 1$$

- $\binom{n}{1}$: # of ways for 1 person to be in 1st place
- $r(n-1)$: Arrangements of remaining participants
- $\binom{n}{2}$: # of ways for 2 people to be in 1st place
- $r(n-2)$: Arrangements of remaining participants
- $\binom{n}{n-1}$: # of ways for all but one person to be in 1st place
- $r(1)$: Last participant
- 1: All participants tied