*This practice plan was created by **Tyler Erb**, a math teacher and coach at Community House Middle School. Tyler created numerous free resources for MATHCOUNTS coaches in his role as the 2021-2022 DoD STEM Ambassador for MATHCOUNTS. Find more resources and information at **dodstem.us**.*

Modular Arithmetic

Warm-Up!

Try these problems before watching the lesson.

Note: *The terms in blue italics commonly appear in competition problems. Make sure Mathletes understand their meaning!*

1. It is currently 4:30 p.m. What time will it be in 2022 minutes?

   We know that every 60 minutes will be an hour, so we divide 2022 by 60 to figure out how many hours will pass. This gives us 33 hours with a remainder of 42 minutes. Next, we know that 24 hours is a full day so we can subtract 24 from 33, giving us 9 extra hours. We add the 9 hours and 42 minutes to 4:30 p.m., giving us 14:12 pm, which does not exist. So, we subtract 12 from the hours and move it to a.m. instead to get 2:12 a.m.

2. It is currently 4:30 p.m. What time will it be in 2022 hours?

   This time we are dealing with 2022 hours, so we can instead divide by 24 because there are 24 hours in a day. We do not care what day it is, only the remaining hours. When 24 divides 2022, we get a remainder of 6 hours. Adding 6 hours to 4:30 we get 10:30 p.m.

3. What is the units ***digit*** of 4^{2022}?

   The units digit is the only thing that matters, so we can look for a pattern. Looking at only the units digit, we get the following: $4^1 = 4$, $4^2 = 16$ (so a units digit of 6), $4^3 = 64$ (so a units digit of 4). We know the pattern will repeat endlessly with the digits alternating from 4 to 6, with 4 for every odd power and 6 for every even power. If we divide 2022 by 2, we find that it goes in evenly, so the units digit is 6.

4. What is the ***remainder*** when 25^{2022} is divided by 6?

   This one is a bit quicker. We can first divide 25 by 6 and find that it has a remainder of 1. This means that no matter what power it is raised to, it will always have a remainder of 1.

5. You have a pile of marbles. When you divide them up into piles of 7, you have 1 left over. When you divide them up into piles of 4, you have 2 marbles left over. If you know you have more than 111 marbles, what is the least number of marbles you can have?

If we start with the 7, we can figure out that the smallest number that is greater than 100 when divided by 7 and has a remainder of 1 is 113. If we continue to add 7, we can create a list of numbers that also fit this criteria: 113, 120, 127, 134, 141, 148, 155, 162… When we divide by 4, we have 2 left over. The first number larger than 100 that fits the criteria is 102. Adding multiples of 4, we can get a list as follows: 114, 118, 122, 126, 130, 134. This tells us that the least number of marbles we can have is 134.

Coach instructions: *After students try the warm-up problems, play the video and have them follow along with the solutions. After watching the video, they may want to go back and try to come up with faster ways to solve the warm-up problems before moving on to the final problem set.*

The Problems

Take a look at the following problems and follow along as they are explained in the video.

6. What is the ***remainder*** of the 2022nd term in the ***Fibonacci sequence*** divided by 3?

Solution in video. Answer: 2.

7. What is the smallest positive value of $x + y$ if $43x + 85y = 2022$ and both x and y are ***integers***?

Solution in video. Answer: 6.

8. What is the units ***digit*** of 2022^{2022}?

Solution in video. Answer: 4.

9. What are the last two ***digits*** of 504^{4044}?

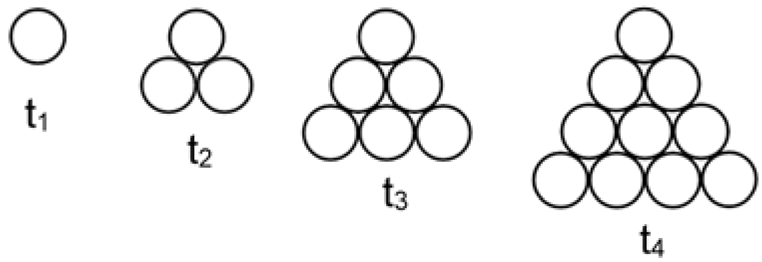
Solution in video. Answer: 56.

Piece It Together

Coach instructions: *After watching the video, give students 15-20 minutes to try the next seven problems.*

Use the skills you practiced in the warm-up and strategies from the video to solve the following problems.

10. A ***triangular number***, t_n, is a number that is created by forming an equilateral triangular grid of points where the first row is 1, and each subsequent row contains one more element than the previous one. The first four terms are shown and are 1, 3, 6, 10. What is the ***remainder*** of the 2022nd number when it is divided by 7?



We are looking for the remainder when dividing by 7, so essentially we are looking at the sequence (mod 7). We start by listing our numbers and then changing them to their (mod 7) equivalent to see if we can find any pattern. To find the next triangular number, add *n* to the previous triangular number. This gives us the following: 1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78, 91, 105, 120… We do not need all of these numbers to realize the pattern, but taking the numbers (mod 7) we have: 1, 3, 6, 3, 1, 0, 0, 1, 3, 6, 3, 1, 0, 0, 1. The pattern repeats after 7 terms. (This works for all primes greater than 2.) If we take 2022 (mod 7), this gives us 6th. The 6th term in the sequence is 0, so our final answer is 0.

11. What is the units ***digit*** of the ***remainder*** when 16^{2022} is divided by 35?

There is no way the problem expects us to actually find 16^{2022}. Even putting the number in our calculator gives an overflow. Instead, we do a bit of case work to look at the powers of 16^n. $16^1 = 16$. I am not going to look at just the units digit because we are dealing with (mod 35). It might not give us the same units digit every time we divide. $16^2 = 256 \equiv 11 \pmod{35}$. $16^3 = 16^2 \times 16 \equiv 11 \times 16 \equiv 176 \equiv 1 \pmod{35}$. Our pattern will then repeat again at 16 and so on. This means our pattern for the units digit is 6, 1, 1. It takes 3 terms to repeat, so we take 2022 (mod 3), which is 0. This means it goes in perfectly, so our final units digit will be the last term in our pattern, or 1.

12. What is the units ***digit*** of the ***remainder*** when 144^{2345} is divided by 35?

The number is divided by 35, so we can take 144 mod 35. This leaves us with 4^{2022}. Looking only at the units digit, we try and find a pattern. $4^1 = 4$; $4^2 = 16$; $4^3 = 64 \equiv 29 \pmod{35}$; $4^4 \equiv 29 \times 4 = 116 \equiv 11 \pmod{35}$; $4^5 \equiv 11 \times 4 = 44 \equiv 9 \pmod{35}$; $4^6 \equiv 9 \times 4 = 36 \equiv 1 \pmod{35}$. Again, we wanted to look at more than the units digit because we were dealing with (mod 35), so the remainder will change depending on what two-digit number we are multiplying by. We see that the pattern repeats every 6 powers. If we take 3333 mod 6, we get a remainder of 5. This tells us that our units digit will have the same remainder as 45, which is 9.

13. What are the last two ***digits*** of the product of 34^{2021} and 37^{2022} divided by 35?

Taking both numbers mod 35, we see that the 34 ≡ −1 (mod 35), so we can treat 34^{2021} as $(-1)^{2021} \equiv -1$ (mod 35). We can rewrite this as 34 if we want, but we will have to multiply it again, so let us leave it as a −1. 37 ≡ 2 (mod 35), so $37^{2022} \equiv 2^{2022}$ (mod 35). We can also use −33 instead of 2, but this isn't entirely helpful when dealing with higher powers. $2^1 = 2$; $2^2 = 4$; $2^3 = 8$; $2^4 = 16$; $2^5 = 32$; $2^6 = 64 \equiv 29$ (mod 35); $2^7 \equiv 29 \times 2 = 58 \equiv 23$ (mod 35); $2^8 \equiv 23 \times 2 = 46 \equiv 11$ (mod 35); $2^9 \equiv 11 \times 2 = 22$ (mod 35); $2^{10} \equiv 22 \times 2 = 44 \equiv 9$ (mod 35); $2^{11} \equiv 9 \times 2 = 18$ (mod 35); $2^{12} \equiv 18 \times 2 = 36 \equiv 1$ (mod 35). We notice the pattern repeats every 12 powers. Notice in the previous problem, it took 6 powers to repeat when dealing with 4*n*. (This pattern does not follow with 8*n*, as it takes 4 powers to repeat, and 3 powers to repeat with 16). We take 2022 mod 12 and get 6. This tells us it will be the same as $2^6 = 64 \equiv 29$ (mod 35). The problem simplifies down to −1 × 29 = −29 ≡ 6 (mod 35). Therefore, our final answer is 6.

14. You and a group of 3 friends find a treasure chest filled with gold coins. You find that if you split the coins up into piles of 4 or 7, there are 2 remaining. If you split the coins into stacks of 9, you find that there is a remainder of 1. You know that there are at least 100 coins. What is the least amount of gold coins possible?

With both of the stacks of 4 and 7 having a remainder of 1, we know that they will also have a remainder of 2 if they were stacks of 28 instead, as 28 is the LCM of the two numbers. The fact that there are at least 100 coins leads us to start our list at 114, which is the smallest number greater than 100 that is 2 more than a multiple of 28. Also, we know that we are looking for a number that is 1 more than a multiple of 9. If the sum of the digits is one more than a multiple of 9, then we know we can stop. We can add multiples of 28 to 114 to create the following list: 114, 142, 170, 198, 226…. And we are done! 226 is one more than a multiple of 9 as the sum of the digits of 226 is 9.

15. There are exactly three ***ordered pairs*** of positive ***integers*** (x, y) that satisfy the equation $12x + 21y = 261$. What is the sum of all the x coordinates of these ***ordered pairs***?

As with most MATHCOUNTS problems, there are multiple ways to solve this one. Here, we will tackle it with Modular Arithmetic, but other ways exist. First, we realize the entire equation is divisible by 3, so we simplify it to $4x + 7y = 87$. Next, we look at our *y* and *x* intercepts. Our *y*-intercept is around 12.5, and our *x*-intercept is 21.75. This will give us a range of values we can use for both *x* and *y*. If we take the entire equation (mod 4), we can simplify this into a one-variable equation. The new equation becomes $3y \equiv 3$ (mod 4). This gives us $y = 1$. Plugging this back into our original equation gives us 20 for *x*. However, the problem told us there were 3 solutions that worked! If we solve our equation for *x*, we get the following: $x = -7/4y + 87/4$. Although this step is unnecessary for solving, it may help us realize that our other possible *y* values must be smaller or larger by a multiple of 4. If $y = 1$ works, so should 5 and 9. Any value greater or less than those three values for *y* will result in *x* values that are negative. We plug in 5 for *y* to get 13 and 9 in for *y* to get 6. The total sum of our *x* values is 20 + 13 + 6 or 39.

16. There is exactly one ***ordered pair*** of positive ***integers*** (x, y) that satisfies the equation $76x + 39y = 881$. What is the sum of x and y?

For this linear equation, the x-intercept is around 11, and the y-intercept is around 22. This means that it might be a tad tedious to try to plug in values 1 to 11 to try to find a possible integer value for y, but it is possible. Instead, we take the entire equation mod 39 to make it a one-variable equation. Again, we want to take the mod of the smaller coefficient to simplify our equation. Otherwise, it becomes harder to deal with. This gives us $37x \equiv 23 \pmod{39}$. However, I do not know the multiplicative inverse of 37 (mod 39). Instead, let us take the remainders a different way. We know that subtracting 39 from 37 will give us −2, which is still equivalent (mod 39). I cannot use $-2x \equiv 23 \pmod{39}$, because this would give us a negative answer and a fraction. Instead, we subtract 39 from 23 to get −16, which is still equivalent (mod 39). $-2x \equiv -16 \pmod{39}$ gives us $x \equiv 8 \pmod{39}$. The problem asks for the sum of x and y, so we plug 8 back in and solve. $76(8) + 39y = 881$, so $39y = 273$, which means y is 7. Thus, $x + y = 8 + 7 =$ 15.

Optional Extension

To extend your understanding and have a little fun with math, try the following activity.

A mogul has decided to buy up land and make a new development with 108 acres of land. The developer wants to have plots that are 2 acres, 5 acres or 7 acres. There must be exactly 20 total plots, and there must be at least one plot of each size. How many total combinations of plots are there?

First, we will set up two equations with the given information: $2x + 5y + 7z = 108$ and $x + y + z = 20$, where x, y and z are the number of plots per their respective size. We also know that there must be at least 1 plot of each size. The max number we can have of any plot would be 18, because there must be 1 each of the other two. Our first step is to eliminate one of the variables. In this case, let us eliminate x by multiplying the bottom equation by −2 and then combining the two. We do not have to eliminate x, but we do have to get rid of at least one variable. This leaves us with $3y + 5z = 68$. From there, we can try and get lucky plugging values in, or we could solve for a variable and try and find integer values. Instead, we use (mod 3). This allows us to solve for one possible solution for z.

$(3y + 5z = 68) \bmod 3 \equiv 2z = 2 \pmod{3}$. Therefore, the smallest possible solution for z is 1. As we just used (mod 3), we know our other solutions must be equivalent to 1 (mod 3). Other solutions are 4, 7, 10, 13 and so on. However, we can stop at 13, because if z is larger than 13, we start to get a non-positive solution for y. Now, we need to check if our values for z will work. Plugging in $z = 1$, we get the following two equations: $2x + 5y = 101$ and $x + y = 19$. If we eliminate x, we get $3y = 63$, or $y = 21$. This can't be a possible solution, because that means x is negative. Plugging in $z = 4$, we get the following two equations: $2x + 5y = 80$ and $x + y = 16$. If we eliminate x, we get $3y = 48$, or $y = 16$. However, if we have $z = 4$ and $y = 16$, then x must be 0, and we need to have a positive whole number of plots for x. If we check when z equals 7, 10 or 13, these all work. Our final answer is 3 possibilities.