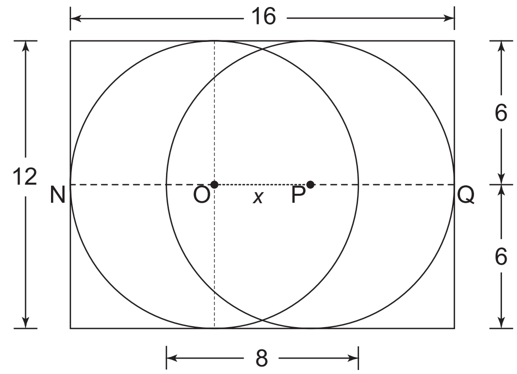
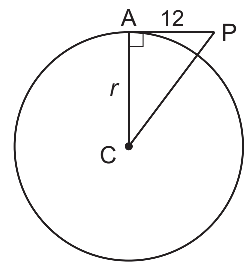
MATHCOUNTS Minis

March 2013 Activity Solutions

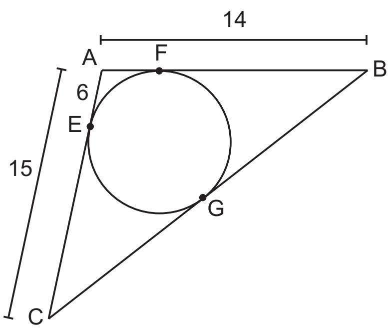
Warm-Up!



1. Based on the information provided in the problem, we've indicated many measurements of the figure here. Since the width of the rectangle is also the diameter of each circle, we know that the radius of each circle is 6 units. Since NO + OP + PQ = 16, with segments NO and PQ being radii, we can write the equation $6 + x + 6 = 16$, when $OP = x$. Simplifying leads to $6 + x + 6 = 16 \rightarrow x + 12 = 16 \rightarrow x =$ **4** units.



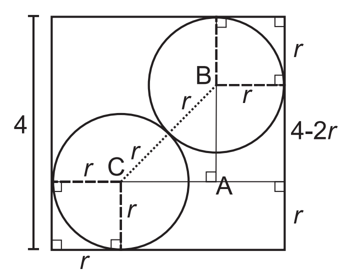
2. Since segment AP is tangent to the circle at A, segment PA will be perpendicular to radius AC. Because the area of the circle is 256π units^2, we can write the following equation and solve for r: $256\pi = \pi r^2 \rightarrow 256 = r^2 \rightarrow r = 16$ units. Using the Pythagorean Theorem with right triangle APC, we now can write the following equation and solve for PC: $(PC)^2 = 12^2 + 16^2 \rightarrow (PC)^2 = 144 + 256 \rightarrow (PC)^2 = 400 \rightarrow PC =$ **20** units.



3. Since AC = AE + EC, we can determine $EC = 15 - 6 = 9$ units. Additionally, from a point outside of the circle, the two segments from that exterior point to the two different points of tangency are equal. Thus, AE = AF, BF = BG and CG = CE. It follows that AF = 6 units, $BF = 14 - 6 = 8$ units, BG = 8 units, CG = 9 units, and finally, $CB = 9 + 8 =$ **17** units.

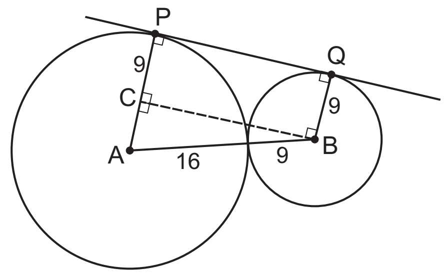
The Problem is solved in the MATHCOUNTS Mini.

Follow-up Problems



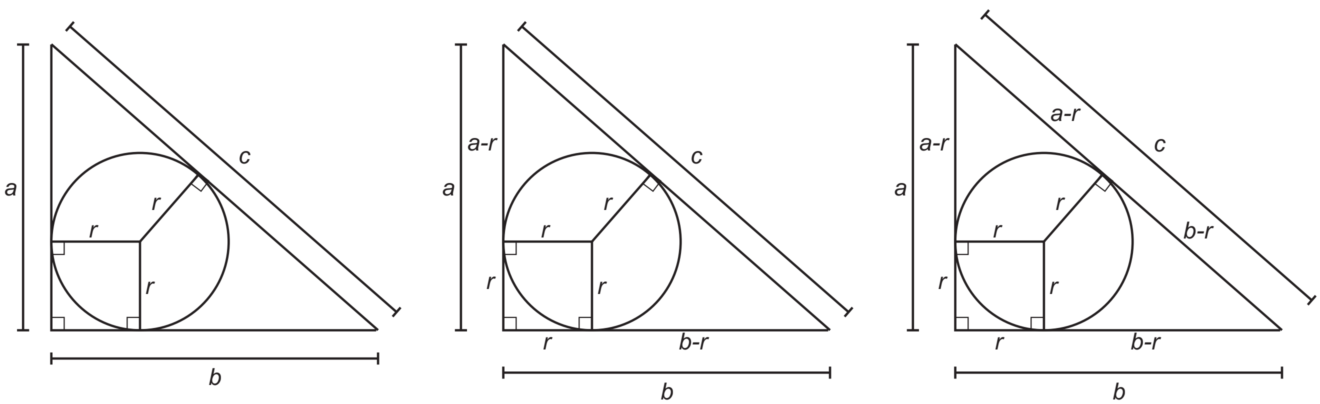
4. In the figure shown here, we have indicated the different radii and right angles that are known from the information given. Additionally, since each side of the square is 4 units, we can see how the right side is divided into segments of lengths r units, r units and $4 - 2r$ units. Thus, $AB = 4 - 2r$. Similarly, $AC = 4 - 2r$. Seeing that $BC = 2r$ and using the Pythagorean Theorem with right triangle ABC, we can write the following equation and solve for r: $(2r)^2 = (4 - 2r)^2 + (4 - 2r)^2 \rightarrow 4r^2 = 16 - 16r + 4r^2 + 16 - 16r + 4r^2 \rightarrow 0 = 4r^2 - 32r + 32 \rightarrow 0 = r^2 - 8r + 8$. Using the Quadratic Formula with $a = 1$, $b = -8$ and $c = 8$, we get

$r = \dfrac{8 \pm \sqrt{(-8)^2 - 4(1)(8)}}{2(1)} = \dfrac{8 \pm \sqrt{64 - 32}}{2} = \dfrac{8 \pm \sqrt{32}}{2} = \dfrac{8 \pm 4\sqrt{2}}{2} = 4 \pm 2\sqrt{2}$. Since $4 + 2\sqrt{2}$ is too large (it's greater than the side of the square), the radius is $\mathbf{4 - 2\sqrt{2}}$ units.



5. In the figure shown here, we have added the segment from B that is perpendicular to radius AP. This segment completes rectangle BCPQ, and now BQ = PC, so PC = 9 units. Radius AP is 16 units, so AC = 16 − 9 = 7 units. When we connect the two centers of the externally tangent circles, we get AB = 16 + 9 = 25 units. Now, using the Pythagorean Theorem with right triangle ABC, we have $25^2 = 7^2 + (BC)^2 \rightarrow 625 = 49 + (BC)^2 \rightarrow (BC)^2 = 576 \rightarrow BC = 24$ units. Because of rectangle BCPQ, we now know PQ = **24** units, too.

6. The series of figures below shows the steps that lead to the third figure. Looking at the third figure, we see that the hypotenuse has a length of c, which is composed of the two lengths $a - r$ and $b - r$. Thus, we can write the following equation and solve for r: $c = (a - r) + (b - r) \rightarrow c = a + b - 2r \rightarrow 2r = a + b - c \rightarrow$ $\mathbf{r = (a + b - c) \div 2}$ units.



7. Using the same right triangle used in problem 6, we can think of its area as (1/2)(*base*)(*height*), or (1/2)ba. However, we also could draw in segments OA, OB and OC from center O. Now we have divided triangle ABC into triangles AOC (shaded), AOB and BOC. The area of triangle AOC is (1/2)(*base*)(*height*), or (1/2)rb, and the areas of triangles AOB and BOC are (1/2)rc and (1/2)ra, respectively. Setting the two representations of the area of triangle ABC equal to one another, we have $(1/2)ba = (1/2)ra + (1/2)rb + (1/2)rc \rightarrow ba = ra + rb + rc \rightarrow ab = r(a + b + c) \rightarrow$ $\mathbf{r = (ab) \div (a + b + c)}$ units.