Activity Sheet for the October, 2009, MATHCOUNTSMini

Warm-Up!

Try these problems before watching the lesson.

1. In how many different ways can three people finish a race if no ties are allowed?
2. In how many different ways can four people finish a race if no ties are allowed?
3. Find a formula for the ways different ways in which n people can finish a race if no ties are allowed.
4. In how many different orders can two people finish a race if ties *are* allowed?

The Problem

Alfred, Brandon and Charles are the three participants in a race. In how many different ways can the three finish if it is possible for two or more participants to finish in a tie? *2009 State Sprint Round* #15

Follow-up Problems

5. Alfred, Brandon and Charles are the three participants in a race. In how many different ways can the three finish if it is possible for two participants to finish in a tie, but not for all three to finish in a tie?
6. Alfred, Brandon, Charles, and David are the four participants in a race. In how many different ways can the four finish if it is possible for two or more participants to finish in a tie?

Further Exploration

In the Warm-Up Problems, you should have found a formula for the number of ways n people can finish a race if no ties are allowed. Is there a similar formula for the number of ways n people can finish a race if ties are allowed?

There isn't a nice, neat formula, but there is a nice, neat method, called *recursion*. A sequence of numbers is the result of a recursion if each number is derived from previous numbers. So, if we say that $f(n)$ is the n^{th} term of the sequence

$$1, 2, 3, 4, 5, 6, 7, 8, \ldots,$$

then we can say that $f(1) = 1$ and $f(n)$ satisfies the recursion $f(n) = f(n-1)+1$ for $n > 1$. This is because the first number is 1, and each number afterwards is 1 plus the previous number.

As our further exploration, we will investigate a sequence called the **Fibonacci numbers**, which are among the most famous examples of recursion.

7. Starting with $1, 1$, make a list of 15 numbers such that each is the sum of the previous two. If we continue this list forever, the numbers in the list are called the **Fibonacci numbers**.
8. Lou is climbing a flight of 4 stairs. With each step, he will climb either 1 or 2 stairs. In how many different ways can he climb the flight of stairs?
9. Answer the previous question for the cases of 5, 6, 7, and 8 stairs. Do you recognize the resulting numbers?
10. Suppose that $f(n)$ is the number of ways Lou can climb n stairs, where n is a positive integer. Explain why $f(n+1) = f(n) + f(n-1)$ for all positive integers n greater than 1.

Note to instructor: The remainder of the exploration is only appropriate for students who have learned binomial coefficients.

The recursion for the races-with-ties problem is considerably more complicated than the recursion for the Fibonacci numbers. Letting $r(n)$ be the number of ways n racers can finish if we allow ties, we have $r(1) = 1$, and

$$r(n) = \binom{n}{1} r(n-1) + \binom{n}{2} r(n-2) + \binom{n}{3} r(n-3) + \cdots + \binom{n}{n-1} r(1) + 1$$

for all integers n greater than 1.

11. Use the recursion to find $r(2)$. Compare it to the answer you found in Problem 4.
12. Use the recursion to find $r(3)$. Compare it to the answer we found in the video.
13. Use the recursion to find $r(4)$. Compare it to the answer you found in Problem 6.
14. (Very hard!) Explain why the recursion works.

Wow! Share Your Thoughts

Have some thoughts about the video? Want to discuss the problems on the Activity Sheet? Visit the MATHCOUNTS Facebook page or the Art of Problem Solving Online Community (www.artofproblemsolving.com).