MATHCOUNTS Minis October 2011 Activity Solutions

Warm-Up!

1a. Using the distributive property we get $(x-1)(x+1) = (x-1)(x) + (x-1)(1) =$ $x^2 - x + x - 1 = \mathbf{x^2 - 1}$.

1b. Using the distributive property we get $(x-2)(x+2) = (x-2)(x) + (x-2)(2) =$ $x^2 - 2x + 2x - 4 = \mathbf{x^2 - 4}$.

1c. Using the distributive property we get $(x-y)(x+y) = (x-y)(x) + (x-y)(y) =$ $x^2 - xy + xy - y^2 = \mathbf{x^2 - y^2}$.

2a. $5^2 - 4^2 = 25 - 16 =$ **9**

2b. $6^2 - 5^2 = 36 - 25 =$ **11**

2c. $7^2 - 6^2 = 49 - 36 =$ **13**

2d. $8^2 - 7^2 = 64 - 49 =$ **15**

2e. It appears that the difference of the squares of two consecutive integers is equal to the sum of the two consecutive integers. To see if this is always true, let's do a little algebra. Let $n - 1$ and n be two consecutive integers. What do we get when we simplify $n^2 - (n-1)^2$? Start by expanding $(n-1)^2$ to get $n^2 - (n^2 - n - n + 1) = n^2 - n^2 + n + n - 1 = n + n - 1 = n + (n - 1)$, which is the sum of the two consecutive integers. Therefore, it is the case that the difference of the squares of two consecutive integers equals the sum of the two consecutive integers.

3. Since we have five consecutive integers, let's let the middle integer be x. Then the five integers in order are $x - 2$, $x - 1$, x, $x + 1$, $x + 2$. So $(x-2) + (x-1) + x + (x+1) + (x+2) = 300$. Simplifying we have $5x = 300 \rightarrow x = 60$. That means the largest of these integers is $x + 2 = 60 + 2 =$ **62**.

4. A prime number is a positive integer and has only two factors, 1 and itself, or −1 and the opposite of itself. Therefore, it must be true that if $(t+4)(t-26)$ is prime, $t + 4 = 1$, $t + 4 = -1$, $t - 26 = 1$ or $t - 26 = -1$. If $t + 4 = 1$, then $t = -3$. If $t + 4 = -1$, then $t = -5$. Neither of these cases produces a positive value for t. If $t - 26 = 1$, then $t = 27$. Substituting this into the original expression, we get $(31)(1) = 31$, and 31 is prime. If $t - 26 = -1$, then $t = 25$. Substituting this into the original expression, we get $29(-1) = -29$, which is not positive. Therefore, the positive integer t for which the product $(t+4)(t-26)$ is prime is **27**.

The Problem is solved in the MATHCOUNTS Mini.

Follow-up Problems

5. Notice that 6,070,807 is 6,070,809 − 2, and 6,070,811 is 6,070,809 + 2. We need to determine which is greater, n^2 or $(n-2)(n+2)$. We know from the video that the product of the difference and the sum of the two integers is the difference of the squares of the two integers. So, $(n-2)(n+2) = n^2 - 4$. Since $n^2 > n^2 - 4$, we can see that $\mathbf{6{,}070{,}809^2}$ is greater.

6. Since 55,556 = 55,555 + 1 we can write $55{,}556^2 = (55{,}555 + 1)^2 =$ $55{,}555^2 + 55{,}555 + 55{,}555 + 1$. Substituting 3,086,358,025 for $55{,}555^2$, we get 3,086,358,025 + (2)(55,555) + 1 = 3,086,358,025 + 111,1110 + 1 = **3,086,469,136**.

7. Notice that $3^8 - 2^6$ can also be written as $(3^4)^2 - (2^3)^2$. From the video we know that the difference of the squares of two integers is the product of the difference and the sum of the two integers. That means $(3^4)^2 - (2^3)^2 = (3^4 - 2^3)(3^4 + 2^3)$. If we simplify this expression, we get $(81 - 8)(81 + 8) =$ **73 × 89** as the prime factorization.

8. Since *a*, *b* and *c* are consecutive positive odd integers, we can write *a* and *c* in terms of *b* as $a = b - 2$, and $c = b + 2$. We know that $b^2 - (b - 2)^2 = 344$, and $(b + 2)^2 - b^2$ is what we are asked to find. Let's start by simplifying $b^2 - (b - 2)^2 = 344$. We have $b^2 - [(b - 2)(b - 2)] = 344 \rightarrow$ $b^2 - (b^2 - 4b + 4) = 344 \rightarrow b^2 - b^2 + 4b - 4 = 344 \rightarrow 4b - 4 = 344 \rightarrow 4b = 348$. Next, let's simplify $(b + 2)^2 - b^2$. We get $(b + 2)(b + 2) - b^2 \rightarrow b^2 + 4b + 4 - b^2 \rightarrow 4b + 4$. Recall we determined that $4b = 348$, so $4b + 4 = 348 + 4 =$ **352**.

9. We are asked to find all pairs of integers *m* and *n* that satisfy $m^2 = n^2 + 105$, which can be rewritten as $m^2 - n^2 = 105$. We know from problem 1 that $m^2 - n^2 = (m - n)(m + n)$. So now we need to find all pairs of integers that have a product of 105. They are 1 and 105, 3 and 35, 5 and 21 and 7 and 15. Notice in each factor pair that we need the smaller number to be *m* minus some quantity *n*, and we need the larger number to be *m* plus some quantity *n*. Since subtracting *n* from *m* yields one factor and adding *n* to *m* yields the other, it must be that *m* will be halfway between the integers in each pair of factors. Midway between 1 and 105 is the average of the two integers. So we have 1 + 105 = 106, and 106/2 = 53. Notice that 1 = 53 − 52, and 105 = 53 + 52. So we have the pair ***m* = 53** and ***n* = 52**. Next, 35 + 3 = 38, and 38/2 = 19. We see that 3 = 19 − 16, and 35 = 19 + 16. Thus, we have the pair ***m* = 19** and ***n* = 16**. Next, 21 + 5 = 26, and 26/2 = 13. Notice that 5 = 13 − 8, and 21 = 13 + 8. That means when ***m* = 13**, ***n* = 8**. Finally, 15 + 7 = 22, and 22/2 = 11. We get 7 = 11 − 4, and 15 = 11 + 4. So, the final pair is ***m* = 11** and ***n* = 4**.

Notice that this method of finding the average works because both factors are odd integers and their sum is divisible by 2. Another method that will work regardless of whether each factor is odd is solving algebraically. We have $m - n = 1$ and $m + n = 105$. When these two equations are added the *n*s cancel and the result is $2m = 106 \rightarrow$ ***m* = 53**. Substituting that in for *m* in either of the original equations yields ***n* = 52**.