MATHCOUNTS® Minis — October 2013 Activity Solutions

Warm-Up!

1. Using the F.O.I.L. method to multiply the binomials, we get $(x + 2)(x - 7) = x^2 + 2x - 7x - 14$. Once we combine like terms, our answer is $\mathbf{x^2 - 5x - 14}$.

2. Again, using the F.O.I.L method, we get $24 - 3y - 8y + y^2$. We then combine like terms to get the result $\mathbf{y^2 - 11y + 24}$.

3. By factoring the polynomial we can determine for what values of r it is true that $r^2 + 3r - 70 = 0$. Factoring the trinomial, the equation can be rewritten as $(r + 10)(r - 7) = 0 \rightarrow r + 10 = 0$ or $r - 7 = 0$. Solving these two equations, we get the real roots $r = \mathbf{-10}$ and $r = \mathbf{7}$.

4. Let's use the same approach we used in the previous problem. Factoring the trinomial, $x^2 - 13x + 40 = 0$ can be rewritten as $(x - 5)(x - 8) = 0 \rightarrow x - 5 = 0$ or $x - 8 = 0$. Solving these two equations, we get the real roots $x = 5$ and $x = 8$. Thus, their sum is $5 + 8 = \mathbf{13}$.

The Problem is solved in the video.

Follow-up Problems

5. As the activity sheet states, this problem appears to be similar to the problem solved in the video, but this quadratic equation has a leading coefficient of 2, not 1. We are told that $2x^2 + bx + c = 0$ when $x = 4$ or $x = -6$, so we can write $(x - 4)(x + 6) = 0$. Expanding the left-hand side of the equation yields $x^2 + 6x - 4x - 24 = 0 \rightarrow x^2 + 2x - 24 = 0$. Since the quadratic equation we were given has a leading coefficient of 2, we need an equivalent expression in the form $2x^2 + bx + c = 0$. Multiplying both sides of the equation by 2 yields $2(x^2 + 2x - 24) = 2(0) \rightarrow 2x^2 + 4x - 48 = 0$. By inspection, we see that $b = 4$ and $c = -48$. Therefore, $b + c = 4 + (-48) = \mathbf{-44}$.

Another approach to solve this problem would be to divide each side of the equation $2x^2 + bx + c = 0$ by 2 to get an equivalent equation in which the trinomial has a leading coefficient of 1. That gives us $x^2 + (b/2)x + c/2 = 0$. Again, since $x = 4$ and $x = -6$ are solutions, we can write $(x - 4)(x + 6) = 0 \rightarrow x^2 + 2x - 24 = 0$. Now, by inspection, we see that $b/2 = 2 \rightarrow b = 4$, and $c/2 = -24 \rightarrow c = -48$. Again, $b + c = 4 - 48 = -44$.

6. We are told that $x = 2$ is a solution to $x^2 + bx + 24 = 0$. That means that $x - 2$ is a factor of $x^2 + 6x + 24$. We also know that the constants in the two binomial factors must have a product of 24. Since $24 \div (-2) = -12$, the other binomial factor must be $x - 12$, and it follows that $x = \mathbf{12}$ is the other root.

7. Using a method similar to the method employed in the video, we can simplify this equation using substitution. Let's rewrite the equation $\sqrt[3]{x^2} - 3\sqrt[3]{x} = 28$ so that the right-hand side equals 0. We have $\sqrt[3]{x^2} - 3\sqrt[3]{x} - 28 = 0$. Recall the property of exponents that states that $(x^a)^b = (x^b)^a$. So, $\sqrt[3]{x^2} = (\sqrt[3]{x})^2$. Now let $y = \sqrt[3]{x}$, and we can rewrite the equation as $y^2 - 3y - 28 = 0$. Factoring, we get $(y - 7)(y + 4) = 0$, and $y - 7 = 0$ or $y + 4 = 0$. So, we have roots $y = 7$, and $y = -4$. Therefore, the values of x that make the given equation true are $7^3 = \mathbf{343}$ and $(-4)^3 = \mathbf{-64}$.

8. We start by subtracting $11t^2$ from both sides of the equation so that the right-hand side of the equation equals zero. We end up with $t^4 - 11t^2 + 18 = 0$. Now we can simplify the problem using substitution. If we let $x = t^2$, then we can rewrite the equation as $x^2 - 11x + 18 = 0$. Factoring, we see that $(x - 9)(x - 2) = 0 \rightarrow x - 9 = 0$ or $x - 2 = 0$. Thus, $x = 9$ and $x = 2$ are roots of this quadratic equation. To determine the roots of the original fourth degree polynomial equation, we substitute 9 and 2 for x in the equation $x = t^2$. We have $9 = t^2 \rightarrow t = \mathbf{\pm 3}$ and $2 = t^2 \rightarrow t = \mathbf{\pm\sqrt{2}}$.

9. First, we rewrite the equation $4^x = 33 \cdot 2^{x-1} - 8$ as $4^x - 33 \cdot 2^{x-1} + 8 = 0$. Since $4 = 2^2$, it follows that $4^x = (2^2)^x = (2^x)^2$. Also, since $2^{x-1} = 2^x \cdot 2^{-1} = 2^x \cdot (1/2)$, we can write $(2^x)^2 - 33 \cdot 2^x \cdot (1/2) + 8 = 0 \rightarrow (2^x)^2 - 2^x \cdot (33/2) + 8 = 0$. Now we can let $y = 2^x$, and rewrite the equation as $y^2 - (33/2)y + 8 = 0$. To eliminate the fraction, we can multiply each side of the equation by 2, to get $2y^2 - 33y + 16 = 0$. Factoring the trinomial, we get $(2y - 1)(y - 16) = 0$. So, $2y - 1 = 0 \rightarrow 2y = 1 \rightarrow y = 1/2$, and $y - 16 = 0 \rightarrow y = 16$ are solutions to this quadratic equation. To solve the original equation, we substitute 1/2 and 16 for y in the equation $y = 2^x$. We have $1/2 = 2^x \rightarrow 2^{-1} = 2^x \rightarrow x = \mathbf{-1}$, and $16 = 2^x \rightarrow 2^4 = 2^x \rightarrow x = \mathbf{4}$.

10. Consider a quadratic equation of the form $ax^2 + bx + c = 0$. We'll factor the given quadratic expressions to see if we can establish a rule to determine the product and sum of the roots of a quadratic equation without solving to find the roots.

(a) $x^2 + 2x - 35 = 0$

In this case, $a = 1$, $b = 2$ and $c = -35$. Factoring, we get $(x + 7)(x - 5) = 0$, and it follows that the roots are $x = -7$ and $x = 5$. The product of the roots is $(-7)(5) = -35$. The sum of the roots is $-7 + 5 = -2$. It looks like the product of the roots is the same as c, and the sum of roots is $-b$. Perhaps the remainder of the quadratics will help us see if, and how, a relates to the product and sum of the roots.

(b) $2x^2 - 5x - 3 = 0$

In this case, $a = 2$, $b = -5$ and $c = -3$. Factoring, we get $(2x + 1)(x - 3) = 0$, and it follows that the roots are $x = -1/2$ and $x = 3$. The product of the roots is $(-1/2)(3) = -3/2$. The sum of the roots is $(-1/2) + 3 = 5/2$. It looks like the product of the roots is the same as c/a, and the sum of roots is $-b/a$.

(c) $2x^2 + 4x - 70 = 0$

In this case, $a = 2$, $b = 4$ and $c = -70$. Notice that the coefficients of this quadratic equation are twice the coefficients of the equation in (a). Factoring, we get $(2x + 14)(x - 5) = 0$, and it follows that the roots are $x = -7$ and $x = 5$. The product of the roots is $(-7)(5) = -35$. The sum of the roots is $-7 + 5 = -2$. It looks like the product of the roots is the same as c/a, and the sum of roots is $-b/a$.

(d) $12x^2 - 11x + 2 = 0$

In this case, $a = 12$, $b = -11$ and $c = 2$. Factoring, we get $(4x - 1)(3x - 2) = 0$, and it follows that the roots are $x = 1/4$ and $x = 2/3$. The product of the roots is $(1/4)(2/3) = 1/6$. The sum of the roots is $1/4 + 2/3 = 11/12$. It looks like the product of the roots is the same as c/a, and the sum of roots is $-b/a$.

Examples (b), (c) and (d) establish a pattern. Looking back at (a), we see that the product of the roots is, in fact, the same as c/a, and the sum of the roots is the same as $-b/a$.