MATHCOUNTS®

Least Common Multiple

Warm-Up!

Try these problems before watching the lesson.

Coach instructions: *Give students 5 to 10 minutes to go through the warm-up problems.*

1. What is the smallest two-digit number that is a multiple of 2 and 3?

   The product of 2 and 3 is 6. If we double it, we get the smallest two-digit multiple or 12.

2. What is the smallest number that is a multiple of 2, 3 and 5?

   If we multiply the 2 · 3 · 5 we get 30.

3. What is the smallest perfect square that is divisible by both 2 and 3?

   If we go through our perfect squares until we get a multiple of 2 and 3, we look at 1, 4, 9, 16, 25 and stop at 36.

4. How many positive integers less than 100 are multiples of 2, 3 and 5?

   In problem 2, we saw the smallest multiple of these three numbers was 30. This means that 60 and 90 are also multiples. Giving us 3 positive integers.

The Problems

Take a look at the following problems and follow along as they are explained in the video.

Coach instructions: *After students try the warm-up problems, play the video and have them follow along with the solutions.*

5. What is the least common multiple of 84 and 63?

   Solution in video. Answer: 252.

6. Mrs. Chandler has a class of 20 kids and she wants to give each student at least one pencil to use at MATHCOUNTS practices. Pencils are sold in packages of 12. If Mrs. Chandler wants to give each student the same number of pencils with no extra leftover, what is the smallest number of packages of pencils she could buy?

   Solution in video. Answer: 5 packages.

7. George Washington High School's marching band members can evenly arrange themselves in rows of 8, 9, 10 or 12. What is the least number of students that could be in the marching band?

   Solution in video. Answer: 360 students.

Piece It Together

Coach instructions: *After watching the video, give students 15 to 20 minutes to try the next six problems.*

Use the skills you practiced in the warm-up and strategies from the video to solve the following problems.

8. What is the least common multiple of 48 and 72?

    The prime factorizations of 48 and 72 are 48 = $2^4 \cdot 3$ and 72 = $2^3 \cdot 3^2$. Taking the highest power for each of the prime factors, we get that LCM (48, 72) = $2^4 \cdot 3^2$ = 144.

9. Let LCM (a, b) be the abbreviation for the least common multiple of a and b. What is LCM (LCM (8, 14), LCM (7, 12))?

    The prime factorizations of the four integers are 8 = 2^3, 14 = $2 \cdot 7$, 7 = 7 and 12 = $2^2 \cdot 3$. This means LCM (8, 14) = $2^3 \cdot 7$ and LCM (7, 12) = $2^2 \cdot 3 \cdot 7$. So the LCM (LCM (8, 14), LCM (7, 12)) = $2^3 \cdot 3 \cdot 7$ = 168. *Note: you could skip a step by considering the problem as LCM (8, 14, 7, 12).*

10. What is the least positive integer divisible by the four smallest odd, positive integer?

    The four smallest positive odd integers are 1, 3, 5 and 7. All of these numbers are prime, so their least common multiple is the product of all four numbers or $1 \cdot 3 \cdot 5 \cdot 7$ = 105.

11. What is the smallest perfect square that is divisible by both 4 and 6?

    The prime factorizations are 4 = 2^2 and 6 = $2 \cdot 3$. This would make LCM (4, 6) = $2^2 \cdot 3$ = 12. This, however, is not a perfect square. Any perfect square will have a prime factorization where each prime is raised to an even power. Looking at the prime factorization of LCM (4, 6) = $2^2 \cdot 3$, we see that 2 is raised to an even power so to get to a perfect square, we just need to multiply by 3. The smallest perfect square that is divisible by both 4 and 6 is therefore $(2^2 \cdot 3) \cdot 3 = 2^2 \cdot 3^2 = 4 \cdot 9$ = 36.

12. Buns are sold in packs of 12. Hamburger patties are sold in packages of 8 and veggie burger patties are sold in packages of 10. At a picnic, both hamburgers and veggie burger sandwiches are served. There is a bun for each patty and no extra of buns, hamburgers or veggie burgers. What is the least number of sandwiches that could be served at the picnic?

    If the picnic wants to serve veggie burgers and hamburgers, then there will need to be at least one package of each. This means there will be 8 + 10 = 18 total patties at a minimum. The least common multiple of 18 = $2 \cdot 3^2$ and 12 = $2^2 \cdot 3$ is $2^2 \cdot 3^2$ = 36. This would means 36 sandwiches could be served — 20 veggie burgers and 16 hamburgers. *Note: the multiples of 12 less than 36 are 12 and 24 which cannot be made with combinations of 8 and 10.*

13. Johnny had a full bag of apple seeds. He found that if he repeatedly removed the apple seeds 2 at a time, 1 seed remained in the bag at the end. Similarly, if he repeatedly removed the seeds 3, 4, 5 or 6 at a time from the full bag, 1 seed remained in the bag at the end. What is the least number of seeds he could have in the full bag?

If Johnny removes, 2, 3, 4, 5 or 6 apple seeds, he has 1 leftover. This means the number of seeds in the bag must be 1 more than a multiple of 2, 3, 4, 5 and 6. The least common multiple of those five integers is $2^2 \cdot 3 \cdot 5 = 60$. So the least number of seeds he could have in his full bag is 60 + 1 = 61 seeds.

Optional Extension

Coach instructions: *Once your students have completed the problems and feel they have a comfortable understanding of the concept, let try this problem. This combines number theory and counting into a multi-step problem.*

To extend your understanding and have a little fun with math, try the following activities.

How many positive integers less than 101 are multiples of 3, 4 or 7?

The smallest positive multiple of 3 is $3 \cdot 1 = 3$ and the largest, less than 101, is $3 \cdot 33 = 99$. This means there are 33 positive multiples of 3 less than 101. Similarly, $4 \cdot 1 = 4$ is the smallest multiple of 4 and the largest we need to consider is $4 \cdot 25 = 100$. So we have 25 positive multiples of 4. Lastly, we have $7 \cdot 1 = 7$ to $7 \cdot 14 = 98$ or 14 multiples of 7. You might think that the answer will be 33 + 25 + 14 = 72, but we would be over counting some numbers.

Since we are looking for multiples of 3, 4 *OR* 7 we have to consider all combinations of these numbers. There are 8 multiples of $3 \cdot 4 = 12$ that have been counted twice, 4 multiples of $3 \cdot 7 = 21$ that have been counted twice and 3 multiples of $4 \cdot 7 = 28$ that have been counted twice. Adjusting our count, we get 72 - 8 - 4 - 3 = 57.

This still isn't our answer however. We counted the number $3 \cdot 4 \cdot 7 = 84$ three times in our original count, then we deducted it three times in adjusting for our over counting. This means it isn't represented any longer. Our final answer is 57 + 1 = 58 positive integers.