MATHCOUNTS®

Faster Arithmetic Methods

Warm-Up!

Try these problems before watching the lesson.

Coach instructions: *These problems are taken from old countdown round problems. They should be solvable in 45 seconds. Challenge students to try to solve them all in 5 minutes or less. They might not be able to do this on the initial try, but the video will present strategies for faster computation to help them see how these can all be solved quickly.*

Note: *The terms in blue italics commonly appear in competition problems. Make sure Mathletes understand their meaning!*

1. What is $40\% \times \frac{2}{3} \times 24 \div 0.8$?

Following the operations in order, the computation would be 40% × 2/3 × 24 ÷ 0.8 = (4/10 × 2/3) × 24 ÷ 0.8 = (4/15 × 24) ÷ 0.8 = 32/5 ÷ 8/10 = 32/5 × 10/8 = 8.

OR

The computation can be simplified by noticing opportunities to cancel out common factors. If we rewrite the equation with all the terms in fraction form, we can cancel the 24 in the numerator with the 3 × 8 in the denominator and also the 10 in the numerator with the 10 in the denominator leaving us with 4 × 2 = 8.

$$\frac{4}{10} \times \frac{2}{3} \times 24 \times \frac{10}{8} = \frac{4 \times 2 \times \cancel{24} \times \cancel{10}}{\cancel{10} \times \cancel{3} \times \cancel{8}} = 4 \times 2 = 8$$

2. What is the value of 1 × 12 + 2 × 11 + 3 × 10 + 4 × 9 + 5 × 8 + 6 × 7 + 7 × 6 + 8 × 5 + 9 × 4 + 10 × 3 + 11 × 2 + 12 × 1?

Notice the symmetry in this problem. Only half of the expression needs to be calculated and then that value can be multiplied by two. In other words, it can be rewritten as 2(1 × 12 + 2 × 11 + 3 × 10 + 4 × 9 + 5 × 8 + 6 × 7) = 2(12 + 22 + 30 + 36 + 40 + 42) = 2(182) = 364.

3. Mac has 25 marbles, of which 20% are red. Thayer has 20 marbles, of which 75% are not red. What is the ***absolute difference*** between the numbers of red marbles they have?

Mac has 0.2 × 25 = 5 red marbles. Thayer has (1 – 0.75) × 20 = 0.25 × 20 = 5 red marbles. Which gives and absolute difference of 5 – 5 = 0. Without doing calculation, you could look at the problem and notice that 20% of 25 is the same thing as 25% of 20 and will result in an absolute difference of zero.

4. What is the value of the sum $\frac{1}{9}+\frac{2}{9}+\frac{3}{9}+\frac{4}{9}+\frac{5}{9}+\frac{6}{9}+\frac{7}{9}+\frac{8}{9}$?

Since all the fractions have the same denominator, you could add up all the numerator values first and the simplify: (1 + 2 + 3 + 4 + 5 + 6 + 7 + 8)/9 = 36/9 = 4.

OR

If you rearrange the order of the terms, you find that you can make four pairs of terms that sum to one: (1/9 + 8/9) + (2/9 + 7/9) + (3/9 + 6/9) + (4/9 + 5/9) = 1 + 1 + 1 + 1 = 4.

5. What is the value of the sum 0.49 + 0.53 + 0.55 + 0.47 + 0.48? Express your answer as a decimal to the nearest hundredth.

One solution is to add the terms in order: (0.49 + 0.53) + 0.55 + 0.47 + 0.48 = (1.02 + 0.55) + 0.47 + 0.48 = (1.57 + 0.47) + 0.48 = 2.04 + 0.48 = 2.52.

OR

For a faster computation, notice that all five terms are close to 0.5. We can think of this as 5 × 0.5 = 2.5 plus or minus the sum of the differences between 0.5 and each term. We have −0.01 + 0.03 + 0.05 − 0.03 − 0.02 = 0.02. The final result will be 2.5 + 0.02 = 2.52.

The Problems

Take a look at the following problems and follow along as they are explained in the video.

Coach instructions: *After students try the warm-up problems, play the video and have them follow along with the solutions. After watching the video, they may want to go back and try to come up with faster ways to solve the warm-up problems before moving on tho the final problem set.*

6. What is the value of the sum $\frac{1}{87}+\frac{2}{87}+\frac{3}{87}+\frac{4}{87}+\cdots+\frac{84}{87}+\frac{85}{87}+\frac{86}{87}$?

Solution in video. Answer: 43.

7. What is the value of 1 × 9 + 2 × 99 + 3 × 999 + 4 × 9999 + 5 × 99999?

Solution in video. Answer: 543195.

8. What is the value of 55 × 33 – 15?

Solution in video. Answer: 1800.

Piece It Together

Use the skills you practiced in the warm-up and strategies from the video to solve the following problems.

Coach instructions: *After watching the video, give students 10 minutes or so to try the next five problems. These problems are from old sprint rounds and should each be solvable in 80 seconds or less, but allow students more time here to try to think through how to use these new strategies.*

9. What is 12 × 37 + 12 × 7 + 12 × 6?

Factoring out 12 from each term, you get 12(37 + 7 + 6) = 12(50) = 600.

10. What is the value of $2 \times 6^3 + 6^2 - 7 \times 6^2$?

Factoring out 6^2 from every term, we get $6^2(2 \times 6 + 1 - 7) = 36(6) = 216$.

11. Audra adds the numbers 2018 and 22, then multiplies the result by 2 and assigns this value to *a*. Beto multiplies the numbers 2018 and 2, then adds 22 to the result and assigns this value to *b*. What is the value of *a* − *b*?

Instead of calculating *a* and then *b*, let's write out the whole numeric expression for $a - b = (2018 + 22) \times 2 - (2018 \times 2 + 22)$. We can distribute the 2 to the terms in the left set of parentheses and distribute the −1 to the second expression in the second set of parentheses to rewrite this expression as $2018 \times 2 + 22 \times 2 - 2018 \times 2 - 22$. Notice the we can cancel $\cancel{2018 \times 2} + 22 \times 2 \cancel{- 2018 \times 2} - 22$ leaving us with $22 \times 2 - 22 =$ 22.

12. What is the value of $(1 + 3 + 5 + \ldots + 2017) - (2 + 4 + 6 + \ldots + 2016)$?

If we distribute the −1 to the terms in the second set of parentheses and use the commutative property to reorder the problem, we can rewrite this expression as $1 - 2 + 3 - 4 + 5 - 6 + \ldots - 2016 + 2017$. Notice that the first term is 1 and every subsequent pair of integers after that add to one: $-2 + 3 = 1$, $-4 + 5 = 1$, … , $-2016 + 2017 = 1$. So the value will be one plus the number of pairs in the series. Since every pair starts with an even number begin at 2 and going to 2016, we will have $2016/2 = 1008$ pairs making the answer $1 + 1008 =$ 1009.

13. What is the greatest ***prime*** factor of $3^7 - 27$?

We can rewrite this expression as $3^7 - 3^3 = 3^3(3^4 - 1) = 3^3((3^2)^2 - 1) = 3^3(9^2 - 1) = 3^3 \cdot 80 = 3^3 \cdot 8 \cdot 10 = 3^3 \cdot 2^4 \cdot 5$. So the greatest prime factor is 5.

Optional Extension

Coach instructions: *If students want to explore these ideas further, they can do one or both of the extensions. The first is a proof of an algorithm they should be familiar with. The second is a more complex factoring problem that will require some algebra background to complete.*

To extend your understanding and have a little fun with math, try the following activities.

Option 1

When asked to multiply two multi-digit numbers without a calculator, many people instinctively compute the value using the standard algorithm—multiplying each digit from one number by each digit from the other number, systematically, and adding up the products. For example:

$$\begin{array}{r} 34 \\ \underline{\times 12} \\ 68 \\ \underline{+34} \\ 3780 \end{array}$$

Can you explain or prove, using properties of multiplication, why this algorithm works? Can you write a general statement or proof for multiplying two arbitrary two-digit numbers AB and CD where A, B, C and D represent digits and A and C are not zero?

$$\boxed{\begin{array}{r} 34 \\ \underline{\times 12} \\ 68 \\ \underline{+34} \\ 3780 \end{array}}$$

Let's arrange the problem horizontally as 34×12. In the first step of the algorithm we deal with the units digit of the second number, 2 here, and multiply it by the first number, 34. Next, we move to the tens digit of the second number, 1 here. We can rewrite 12 as $10 + 2$ to represent this and then distribute the 34 to get $34 \times (10 + 2) = 34 \times 10 + 34 \times 2 = 340 + 68 = 3780$. Notice that after distributing and computing the two multiplication terms, we arrive at the addition of the same numbers as in the standard algorithm.

100s	10s	1s
	A	B
×	C	D
	A·D	B·D
+A·C	C·B	
A·C	A·D + C·B	B·D

To prove this algorithm for the general case of AB × CD, let's write the two-digit number AB as 10A + B and the two-digit number CD as 10C + D to establish A and C as digits in the tens place and B and D as digits in the units place so we can multiply. Multiplying, we use F.O.I.L to get (10A + B)(10C + D) = 100A·C +10A·D + 10B·C + B·D = 100A·C +10(A·D + B·C) + B·D. If we compare this to the calculation of AB × CD found using the standard algorithm shown to the left, we see that the hundreds, tens and units digits match the result from F.O.I.L. method.

Option 2

Using factoring, find the value of the following expression:

$$\frac{2017^2 + 11(2017) - 42}{2014}$$

Notice (1) two occurrences of 2017 in the numerator, (2) the trinomial configuration of the expression in the numerator and (3) the denominator value is close to 2017. Let's look just at the numerator first. If we let x = 2017 and rewrite the expression as $x^2 + 11x - 42$, we can see the opportunity for factoring. The expression $x^2 + 11x - 42$ can be factored into the product of two binomials, $x - 3$ and $x + 14$. Plugging 2017 back in for x, we can rewrite the numerator as (2017 – 3)(2017 + 14). The first term, 2017 – 3, is equivalent to the denominator, which is 2014. Canceling out, leaves us with 2017 + 14 = 2031.