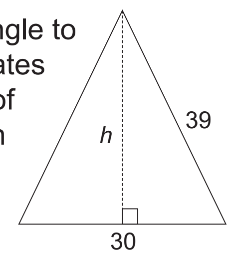
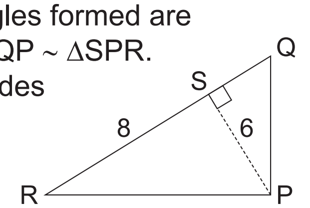
MATHCOUNTS Minis November 2012 Activity Solutions

Warm-Up!

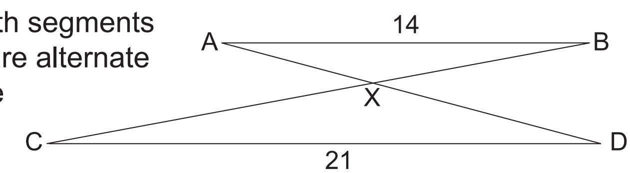
1. Since BA:AC = 3:2, it follows that AC = $\frac{2}{5}$BC. Therefore, AC = $\frac{2}{5} \times 45$ = **18**.



2. We are told the triangle has base length 30, so we just need the height of the triangle to determine its area. The altitude of this triangle drawn from the vertex to the base creates two congruent right triangles, as shown. We know that the length of the hypotenuse of each of these triangles is 39, and the length of the shorter leg is $\frac{1}{2} \times 30 = 15$. We can now use the Pythagorean Theorem to find the height, h, of the isosceles triangle. We have $15^2 + h^2 = 39^2 \rightarrow 225 + h^2 = 1521 \rightarrow h^2 = 1296 \rightarrow h = 36$. Therefore, the area of the isosceles triangle, in square units, is $\frac{1}{2} \times 30 \times 36$ = **540**.



3. Segment PS is an altitude of ΔPQR drawn perpendicular to the hypotenuse, as shown. When an altitude is drawn to the hypotenuse of a right triangle, the two triangles formed are similar to each other and to the original right triangle. Therefore, ΔPQR ~ ΔSQP ~ ΔSPR. Since these triangles are similar, the ratios of the lengths of corresponding sides are equal. So we can write the proportion PS/SR = QP/PR. We are told that PS = 6 and SR = 8, which means PR = 10 (side lengths are a multiple of the Pythagorean Triple 3-4-5). Substituting these values and cross-multiplying yields 6/8 = PQ/10 → 8(PQ) = 6 × 10 → PQ = 60/8 = **15/2**.



4. Parallel segments AB and CD are shown here, with segments AD and BC intersecting at X. Angles BAX and CDX are alternate interior angles, which means they are congruent. The same is true for ∠ABX and angle ∠DCX. Therefore, ΔABX ~ Δ DCX since two triangles are similar if two angles of one triangle are equal in measure to two angles of another triangle (Angle-Angle Similarity). So, we can write the proportion AB/DC = AX/DX. We are told that AB = 14 and CD = 21. We also know what AD = 20, so it follows that DX = 20 − AX. Substituting and cross-multiplying, we get 14/21 = AX/(20 − AX) → 14(20 − AX) = 21(AX) → 280 − 14(AX) = 21(AX) → 280 = 35(AX) → AX = **8**.

The Problem is solved in the MATHCOUNTS Mini.

Follow-up Problems

5. Notice that triangles ABC and ADE share an angle, and each triangle has another angle that measures 90 degrees. Therefore, ΔABC ~ ΔADE. That means the ratio of corresponding sides is proportional. We can write the proportion AE/AD = AC/AB. We are told that AE = 11, AC = 35 and AB = 10 + 11 = 21. Substituting and cross-multiplying yields 11/AD = 35/21 → 35(AD) = 11 × 21 → 35(AD) = 231 → AD = **33/5**.

6. In right triangle MBX, segment MY is an altitude drawn to the hypotenuse. That means $\Delta YBM \sim \Delta YMX$. We can set up the proportion MY/YX = BY/MY. We are told that the area of square WXYZ is 144 units^2. Therefore, the square must have side length $\sqrt{144} = 12$. Also, since M is the midpoint of side YZ, it follows that the YM = 6. Substituting these values into the proportion and cross-multiplying, we get $6/12 = BY/6 \rightarrow 12(BY) = 6 \times 6 \rightarrow 12(BY) = 36 \rightarrow BY = 3$. Since segments WZ and XB are parallel, $m\angle ZAM = m\angle YBM$ as they are alternate interior angles. In addition $m\angle AZM = m\angle BYM = 90°$. We also know that ZM = MY. Therefore, by the Angle-Angle-Side Theorem $\Delta AZM \cong \Delta BYM$, and BY = AZ = **3**.

7. Triangle BCD is a 30-60-90 right triangle with a shorter leg of length 6. Based on properties of 30-60-90 right triangles, segment BC, the longer leg, has length $6\sqrt{3}$. Since M is the midpoint of segment AD, $MD = 6\sqrt{3} \div 2 = 3\sqrt{3}$. For right triangle CDM, we know CD = 6 and $DM = 3\sqrt{3}$, so we can use the Pythagorean Theorem to determine CM. We have $CM^2 = 6^2 + (3\sqrt{3})^2 \rightarrow CM = \sqrt{(36 + 27)} \rightarrow CM = \sqrt{(63)} \rightarrow CM = 3\sqrt{7}$. If $m\angle DBC = 30°$, then $m\angle BDA = 30°$ because they are alternate interior angles. Also $m\angle CKB = m\angle MKD$ since they are vertical angles. That means $\Delta CKB \sim \Delta MKD$, and BC/DM = CK/MK. Substituting and simplifying BC/DM, we have 2/1 = CK/MK, which means $MK = \frac{1}{3}CM \rightarrow MK = \frac{1}{3} \times 3\sqrt{7} \rightarrow MK = \sqrt{\mathbf{7}}$.

8. We are told that DE = 2EC, which means that DE/EC = 2/1, and $DE = \frac{2}{3}DC$. Since AB = DC, it follows that $DE = \frac{2}{3}AB$, and DE/AB = 2/3. Because segments AB and DC are each perpendicular to segment BC, it follows that segment AB and segment CD (or segment DE) are parallel. Thus, $m\angle BAF = m\angle DEF$, and $m\angle FDE = m\angle ABF$ because they are pairs of alternate interior angles. By Angle-Angle Similarity, we have $\Delta ABF \sim \Delta EDF$. Notice that segment BG is an altitude of ΔABF, and segment CG is the corresponding altitude of ΔEDF. Therefore, CG/BG = 2/3 and $BG = \frac{3}{5}BC$. Right triangles BGF and BCD are also similar (Angle-Angle Similarity using the right angles and $\angle FBG$ in each triangle), which means that BC/DC = BG/FG. Substituting and cross-multiplying yields $BC/20 = (\frac{3}{5}BC)/FG \rightarrow BC \times FG = 20(\frac{3}{5}BC) \rightarrow FG =$ **12**.

9. Let's first find an expression to represent the length of segment PX. Since segments AD and PQ are parallel, we know that $m\angle ADB = m\angle PXB$. Notice also that $m\angle ABD = m\angle PBX$. Therefore, $\Delta ABD \sim \Delta PBX$, and AB/PB = AD/PX. Since AB = 6, and P is the midpoint of segment AB, it follows that AP = PB = 3. Substituting and cross-multiplying, we have $6/3 = AD/PX \rightarrow 6(PX) = 3(AD) \rightarrow PX = \frac{1}{2}AD$. Now let's find an expression to represent the length of segment YR. Since segments PR and BC are parallel, we know that $m\angle DYR = m\angle DBC$. Notice also that $m\angle YDR = m\angle BDC$. Therefore, $\Delta YRD \sim \Delta BCD$, and CD/RD = BC/YR. We are told that CD = 8. We know that RC = PB = 3 since quadrilateral PBRC is a parallelogram. It follows, then, that RD = 8 − 3 = 5. Now, substituting these values and cross-multiplying yields $8/5 = BC/YR \rightarrow 8(YR) = 5(BC) \rightarrow YR = \frac{5}{8}BC$. So, $PX/YR = (\frac{1}{2}AD)/(\frac{5}{8}BC)$. But we are told that AD = BC, so we have $PX/YR = (\frac{1}{2})/(\frac{5}{8}) = \frac{1}{2} \times \frac{8}{5} =$ **4/5**.