MATHCOUNTS Minis December 2010 Activity Solutions

Warm-Up!

1. There are two possible outcomes for the first flip (H,T). For each of these, there are two possible outcomes for the second flip; this is a total of four sequences (HH, HT, TH, TT). For each of these, there are two possible outcomes for the third flip; this is a total of eight sequences. Repeating this multiplication by 2, we can see that the total number of possible 7-flip sequences is 2^7 = **128**.

2. Let's rewrite the set as {M, L, H, B, N}. The empty set is 1 subset. There are 5 subsets of one element (or person) each: {M}, {L}, {H}, {B}, {N}. How many two-element subsets are there? Or, how many groups of two can be selected from a group of 5? The answer can be represented as "five choose two" or ${}_5C_2 = (5!)/((2!)(3!)) = 10$. Similarly, how many three-element subsets are there? This will also be 10 since for every two-element subset we included, there was a complementary three-element subset we left out. (In other words, when we counted {M, L} as a two-element subset, we simultaneously found {H, B, N} as a three-element subset.) Similarly, there are 5 subsets of four elements (to complement the 5 subsets of one element) and 1 subset of five elements (to complement the 1 empty set). The total is 1 + 5 + 10 + 10 + 5 + 1 = **32** subsets.

3a. Dividing 100 (the smallest possible positive three-digit integer) by 7, we see the answer is a little more than 14. Therefore 15 × 7 is the smallest positive three-digit multiple of 7. Dividing 999 (the largest possible positive three-digit integer) by 7, we see the answer is a little more than 142, so the largest positive three-digit multiple of 7 is 142 × 7. Using 15 through 142, there are 142 – 14 = **128** positive three-digit multiples of 7.

3b. There are 999 – 99 = 900 positive three-digit integers. Therefore, 900 – 128 = **772** of them are *not* multiples of 7.

4. The positive five-digit integers are 10,000 through 99,999. There are 99,999 – 9999 = 90,000 of them. Let's count how many of them do *not* have any zeros. The first digit could be any digit from 1 through 9. Similarly, each of the four digits after the first is limited to those nine options since we don't want any zeros in our number. This means there are 9 × 9 × 9 × 9 × 9 = 9^5 = 59,049 positive five-digit integers that do *not* have any zeros. Therefore, there are 90,000 – 59,049 = **30,951** positive five-digit integers that have *at least one* zero.

5. Rather than finding the numbers that are *not* perfect squares, let's figure out which numbers *are* perfect squares within the range. The perfect square 12^2 = 144 is just too small. The first perfect square within the range is 13^2 = 169. Similarly, 19^2 = 361 is just too big, but 18^2 = 324 is within the range. Therefore, the perfect squares 13^2 through 18^2 are within the range. That's 18 – 12 = 6 perfect squares. The range only includes the numbers *between* 150 and 350, so that's 349 – 150 = 199 numbers. We've determined 6 of them *are* perfect squares, so 199 – 6 = **193** are *not* perfect squares.

The Problem is solved in the MATHCOUNTS Mini.

Follow-up Problems

6. A subset of the set {M, A, T, H, C, O, U, R, S, E} may contain 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 or 10 letters. As with the problem presented in the MATHCOUNTS Mini, consider that each of the 10 letters must choose whether to be in the subset or not. That means there are a total of

2^{10} = 1024 subsets. To determine the number of subsets with at least two letters we need to exclude the empty set and those subsets containing only one letter. That includes ${}_{10}C_0 + {}_{10}C_1 = 1 + 10 = 11$ subsets. Therefore, the number of subsets that contain at least two letters is 1024 − 11 = **1013** subsets.

7. There are a total of 999 − 99 = 900 positive three-digit numbers. If exactly one digit must be zero it can only be the tens digit or the units digit. So we can select which digit is zero in 2 ways. The remaining two digits cannot be zero, therefore, they can be chosen in 9 × 9 different ways. The total number of positive three-digit numbers that contain exactly one zero is then 2 × 9 × 9 = **162**.

8. With no restrictions, each letter in the five-letter "word" can be one of four letters, so there are $4 \times 4 \times 4 \times 4 \times 4 = 4^5 = 1024$ possible five-letter words. Let's see how many of these words have *no* vowels. There would be only two choices for each letter (C or G), so there are $2 \times 2 \times 2 \times 2 \times 2 = 2^5 = 32$ possible five-letter words. This means there are 1024 − 32 = **992** five-letter words with at least one vowel.

9. Let X represent the units digit which makes the thousands digit 2X. Since 0 × 2 = 0 and the first digit cannot be zero we can exclude 0 from the possible values of X. If X = 5 we have 2(5) = 10, a two-digit number and thus not a candidate for the first digit. For this reason X cannot have a value greater than 4. Thus the possible values of X that would result in a positive four-digit number are 1, 2, 3 and 4. In each of these four cases the tens digit and hundreds digit can be any of the ten numerals 0 to 9, and the thousands digit is pre-determined by the units digit. It follows that the there are 1 × 10 × 10 × 4 = **400** positive four-digit numbers such that the thousands digit is double the units digit.

10. With 9 students there are a total of 9! = 362,880 different ways to seat them in a row. There are many seating arrangements with at least 2 girls next to each other. Consider, instead, the number of arrangements where *no* girls are seated next to each other. That only can occur if the seats are arranged G-B-G-B-G-B-G-B-G where G is a girl and B is a boy. Since there are a total of 5 girls in the class they can be seated in 5! different orders. The four boys in the class can be seated in 4! different orders. Thus there are (5!)(4!) = 120 × 24 = 2,880 different possible seating arrangements in which two girls are *not* seated next to each other. If we exclude these arrangements from the total number of seating arrangements, we see that there are 362,880 − 2880 = **360,000** ways the students can be seated such that at least two girls are seated next to each other.

11. There are twenty-six letters in the alphabet so the first letter in the sequence can be chosen in 26 different ways. If no two adjacent letters can be the same that leaves only 25 different choices for the second letter in the sequence. Since the third letter cannot be the same as the second letter but may be the same as the first letter, there are again 25 choices for this letter. The same is true for the fourth and fifth letters in the sequence. So there are 26 × 25 × 25 × 25 × 25 = **10,156,250** sequences of five letters in which no two adjacent letters are the same.

12. There are 9,999 − 999 = 9,000 positive four-digit numbers. Rather than counting the amount of four-digit numbers with a repeated digit, we'll count the amount of four-digit numbers with four distinct digits. There are only 9 ways to select the first digit since it cannot be zero. But the second digit cannot be the same as the first which leaves only 9 ways to select that digit (since it may be zero). If follows that there are 8 ways to choose the third digit since it cannot be the

same as the first or second digits. Finally there are 7 ways to choose the fourth digit since it cannot be the same as the first, second or third digits. Therefore, there are a total of $9 \times 9 \times 8 \times 7 = 4{,}536$ positive four-digit numbers that have no digit repeated. That means there are $9{,}000 - 4{,}536 = 4{,}464$ positive four-digit numbers in which a digit is repeated. That represents 4,464/9,000 = 0.496 = **49.6%** of all positive four-digit numbers.

Further Exploration

13. Suppose a set has *n* items. Recall from the MATHCOUNTS Mini presentation the table used to count the number of subsets of various sized sets.

# terms in subset ▶	**0**	**1**	**2**	**3**	**4**	**Total**
{M}	1	1	–	–	–	**2**
{M, A}	1	2	1	–	–	**4**
{M, A, T}	1	3	3	1	–	**8**
{M, A, T, H}	1	4	6	4	1	**16**

Given a set of *n* items, to determine the number of subsets containing one term we essentially are calculating ${}_nC_1$. Similarly, to determine the number of subsets with two, three, ..., *n* items we calculate ${}_nC_2, {}_nC_3, \ldots, {}_nC_n$. For all subsets there is only one subset with zero terms, the empty set which corresponds to ${}_nC_0 = 1$. When we sum the number of items contained in all the subsets of a set of *n* items we see that the total is equivalent to 2^n.

As presented in the Mini, there is another way to think of the creation of subsets of a set of *n* items. Let *S* be a set of *n* items. When creating a subset of *S* each term in the set, $s_1, s_2, \ldots., s_n$ has two choices, to be in the subset, or not to be in the subset.

$$\underset{s_1}{\underline{2}} \times \underset{s_2}{\underline{2}} \times \underset{s_3}{\underline{2}} \times \underset{s_4,}{\underline{2}} \times \underset{\ldots,}{\ldots} \times \underset{s_n}{\underline{2}} = 2^n$$