

# MATHCOUNTS® Problem of the Week Archive

## Fall Leaves – October 2, 2023

### Problems & Solutions

It's Autumn, and that means leaves are falling! Amber's house occupies  $2650 \text{ ft}^2$  in the middle of a 50-foot by 80-foot parcel of land, and the leaves that fall cover the ground all around her house. If it takes Amber's family 4.5 hours to rake all the leaves, how many square feet of leaves did they rake each hour?

*The entire parcel of land on which Amber's house sits measures  $50 \times 80 = 4000 \text{ ft}^2$ . Since leaves cover the ground all around the house, it follows that the family raked  $4000 - 2650 = 1350 \text{ ft}^2$  of leaves in 4.5 hours. That means each hour they raked  $1350/4.5 = 300 \text{ ft}^2$  of leaves.*

Last year, when Amber's family raked the leaves, they used one bag per 100 square feet raked. This year, since there were more fallen leaves, they ended up using 50% more bags. How many more bags did they use this year?

*Let's assume that they raked  $1350 \text{ ft}^2$  of fallen leaves last year, and used one bag per 100 square feet raked. Since  $1350/100 = 13.5$ , they would have used 14 bags. Since they used 50% more bags than they did last year, this year they must have used an additional  $0.5 \times 14 = 7$  bags, or 21 bags total.*

Amber especially loves this time of year, when the leaves on the trees in her yard begin changing color to display an array of vibrant colors. There are yellow hickory tree leaves, orange maple tree leaves, purple cherry tree leaves, yellow ash tree leaves and scarlet dogwood tree leaves. Before raking, Amber collected some of the fallen leaves for a craft project. If the leaves Amber collected include at least one leaf from each type of tree, at least two purple leaves and at least three yellow leaves, how many such collections of 10 leaves are possible?

*From the information given, we know that five of the leaves Amber collected were a yellow hickory, a orange maple, a purple cherry, a yellow ash and a scarlet dogwood, and these five leaves include one purple and two yellow leaves. We also know that her collection must include at least one more purple cherry leaf. We now know that any such collection of 10 leaves must include the six leaves we've already identified. So, we need only focus on the four leaves needed to complete a collection of 10 leaves, noting that at least one of the four leaves must be yellow.*

*While we could count how many groups of four leaves that contain at least one yellow leaf, it might be easier to subtract the number of groups of four that **do not** include any yellow leaves from the total number of groups of four leaves. We can do this quickly using the "stars and bars" technique. In this case, we have 4 stars, representing the four leaves, and 4 bars used to separate them into five bins (one for each type of leaf). For example,  $* | * | * | * |$  shows a star in each of the first four bins, depicting a group made up of one yellow hickory leaf, one orange maple, one purple cherry and one yellow ash. Similarly,  $* | * | | | * *$  shows a star in the first two bins and two stars in the fifth bin, depicting a group made up of one yellow hickory leaf, one orange maple and two scarlet dogwoods.*

*Now, to find the total number of groups of four leaves, we need to find the number of permutations of  $* * * * | | |$  (4 stars and 4 bars). Recall that the number of permutations of  $m$  objects with  $n$  objects of one kind and  $m - n$  objects of another kind is  $m!/(n!(m - n)!)$ . So, the total number of groups of four leaves is  $8!/(4! \times 4!) = (8 \times 7 \times 6 \times 5)/(4 \times 3 \times 2 \times 1) = 70$  groups. Next, if we want to count only groups of four that exclude yellow leaves, there are three types of leaves to consider, or three bins with only 2 bars*

needed to separate them. So, to determine the number of groups of four that do not include any yellow leaves, we need to find the number of permutations of \* \* \* \* | | (4 stars and 2 bars). Doing so, we see that there are  $6!/(4! \times 2!) = (6 \times 5)/(2 \times 1) = 15$  groups of four that do not include any yellow leaves. Subtracting, we get  $70 - 15 = 55$  of the collections of 10 leaves described.

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